Branched transportation and singularities of Sobolev maps between manifolds
Part II : Sobolev spaces and topology

Fabrice Bethuel

Laboratoire Jacques-Louis Lions,
Université Pierre et Marie Curie – Paris 6

NONLINEAR FUNCTION SPACES IN MATHEMATICS AND PHYSICAL SCIENCES,
LYON, 14-18 DECEMBER 2015

Topology in $W^{1,p}(\mathscr{M},\mathscr{N})$

So far, we have worked only with continuous maps. Several questions with a topological flavor may however be addressed in Sobolev classes. For instance

$(\mathscr{P}1)$ **Can one define homotopy classes in $W^{1,p}(\mathscr{M},\mathscr{N})$?**

We have seen also other problems:

$(\mathscr{P}2)$ **what about weak limits in Sobolev classes ?**

$(\mathscr{P}3)$ **Can one define liftings in Sobolev classes ?**

A central tool in all of these problemes is the approximation problem:

$(\mathscr{P}4)$ **Can one approximate maps in Sobolev classes by smooth maps, or with prescribed types of singularities ?**

Homotopy classes in Sobolev spaces

Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{N}$ embedded in $\mathbb{R}^\ell$. Recall that

$$W^{1,p}(\mathcal{M},\mathcal{N}) = \{u \in W^{1,p}(\mathcal{M},\mathbb{R}^\ell),\ u(x) \in \mathcal{N} \text{ for a.e } x \in \mathcal{M}\}.$$

Using an approximation argument, one may show that **homotopy classes are well-defined in** $W^{1,p}(\mathcal{M},\mathcal{N})$ in the case $p \geq m = \dim \mathcal{M}$. As a a matter of fact, we have:

Theorem (Schoen-Uhlenbeck)

if $p \geq m$, then $C^\infty(\mathcal{M},\mathcal{N})$ is dense in $W^{1,p}(\mathcal{M},\mathcal{N})$.

Proof of the Schoen and Uhlenbeck theorem

The case $p > \dim \mathscr{M}$. By Sobolev embedding

$$W^{1,p}(\mathscr{M},\mathscr{N}) \hookrightarrow C^0(\mathscr{M},\mathscr{N})$$

For $u \in W^{1,p}(\mathscr{M},\mathscr{N})$ consider

$$u_\varepsilon = \varphi_\varepsilon \star u \text{ with } \varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon^m}\varphi\left(\frac{\cdot}{\varepsilon}\right) \text{ standard molifier,}$$

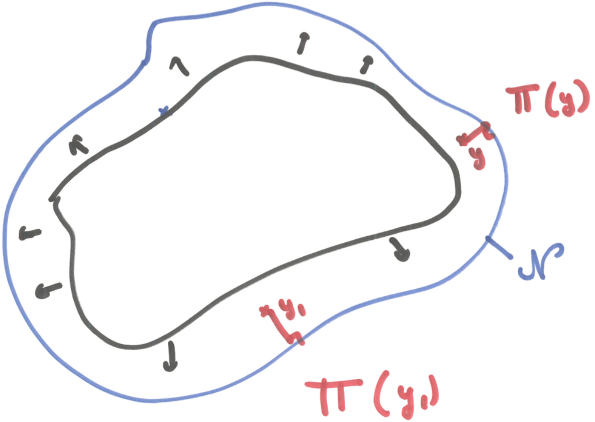
so that $u_\varepsilon \to u$ uniformly and hence

$$\mathrm{dist}(\mathscr{N}, u_\varepsilon(x)) \to 0 \text{ as } \varepsilon \to 0. \tag{1}$$

and one obtains, for π nearest point projection onto $\mathscr{N}$

$$C^\infty(\mathscr{M},\mathscr{N}) \ni \Pi \circ u_\varepsilon \to u \text{ in } W^{1,p} \text{ as} \varepsilon \to 0.$$

The limiting case $p = \dim \mathscr{M}$. The argument may be adapted. Convergence (1) remains true, but with a different argument, and hence the conclusion.



Nearest point projection

Homotopy classes for $p \geq m$

If $p > m$, maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ are continuous, so that homotopy classes are well-defined in $W^{1,p}(\mathcal{M},\mathcal{N})$.

In the limiting case $p = m$, using the previous approximation scheme one may show that all approximating maps are in the same homotopy class, defining hence hence homotopy classes in $W^{1,m}(\mathcal{M},\mathcal{N})$.

Homotopy classes are conserved under weak convergence if $p > m$, that means:

if $u_{n\,n\in\mathbb{N}}$ is a sequence of maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ that are all homotopic and if

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\mathcal{M},\mathcal{N}) \Longrightarrow u \text{ is in the same homotopy class.}$$

This is no longer true in the limiting case $p = m$, due to **the bubbling phenomenon**

Bubbling sequences

Assume that $\pi_m(\mathcal{N}) \neq \emptyset$ and consider a map $\chi \in C^0_{\mathrm{S}}(\mathbb{B}^m, \mathcal{N})$ with a non trivial homotopy class: We extend it to $\mathbb{R}^m$ setting

$$\chi(x) = \mathrm{S} \text{ for } x \in \mathbb{R}^m \setminus \mathbb{B}^m.$$

Next consider the scaled map $\chi_r \in C^0_{\mathrm{S}}(\mathbb{D}^2, \mathbb{S}^2)$ defined by

$$\chi_r(x) = \chi(\frac{x}{r}) \text{ if } |x| \leq r \text{ and } \chi_r(x) = \mathrm{S} \text{ otherwise.}$$

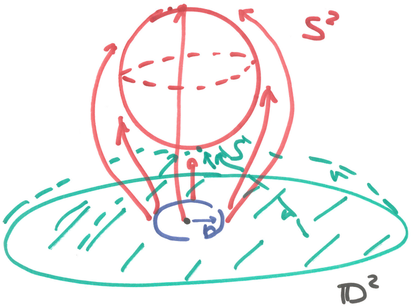
The Dirichlet energy is scale invariant so that

$$\int |\nabla \chi_r|^m = \int |\nabla \chi|^m$$

and χ_r and χ are in the same homotopy However

$$\chi_r \rightharpoonup \mathrm{S}, \text{ as } r \to 0,$$

Since constants have trivial homotopy class the homotopy class not conserved in the weak limit. in the limiting case $p = m \in \mathbb{N}^*$.



The case $p = 2$, $\mathcal{N} = \mathbb{S}^2$

bubbles for the sphere $\mathbb{S}^2$

In this example, let $\mathrm{S} = (0, 0-1)$ be the South pole of $\mathbb{S}^2$ and consider the set

$$C^0_{\mathrm{S}}(\mathbb{D}^2, \mathbb{S}^2) = \{u \in C^0(\mathbb{D}^2, S^2), u = S \text{ on } \partial D^2\}.$$

Homotopy classes in $C^0_{\mathrm{S}}(\mathbb{D}^2, \mathbb{S}^2)$ are labelled by an integer $d \in \mathbb{Z}$, the degree of the map. For instance, let

$$\chi(x_1, x_2) = (x_1 f(r), x_2 f(r), g(r)) \text{ with } r = \sqrt{x_1^2 + x_2^2},\ r^2 f^2(r) + g^2(r) = 1,$$

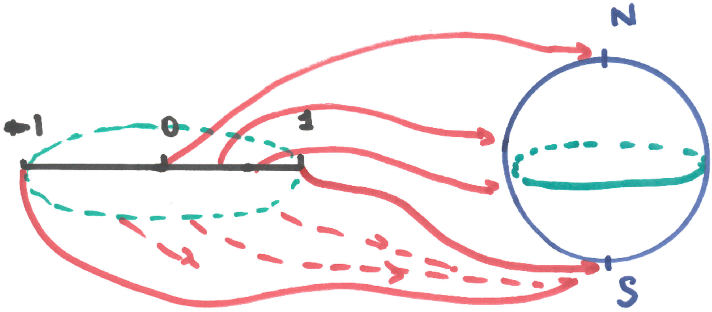
with f and g smooth such that

$$\begin{cases} f(0) = f(1) = 0,\ 0 \le rf(r) \le 1 \text{ for any } r \in [0,1] \\ -1 \le g \le 1 \text{ and } g \text{ decreases from } g(0) = 1 \text{ to } g(1) = -1. \end{cases}$$

Then

$$\deg \chi = 1,$$

whereas the constant map $u = \mathrm{S}$ has degree zero.


-1
0
1
N
S

Infima of energies in homotopy classes

Consider the p Dirichlet energy defined on $W^{1,p}(\mathcal{M},\mathcal{N})$ by

$$E_p(u)=\int_{\mathcal{M}}|\nabla u|^p \mathrm{d}x$$

A natural question is

($\mathscr{P}$5) For $p \geq m$ Is E_p achieved in homotopy classes?

The answer is **YES** if $p > m$. It suffices to invoke thedirect methods in Calculus of variation

The limiting case $p = m$ is fare more subtle in view of the bubbling phenomena. Results are of various nature and will not be discussed here.

estimates of infima of energies in homotopy classes

We are interested here in the value of the number, for a given homotopy class $[\![v]\!]$

$$\kappa_p([\![v]\!]) = \inf\left\{E_p(u), u \in C^1(\mathcal{M},\mathcal{N}) u \in [\![v]\!]\right\}$$

of the energy among all maps with the same energy as v. We will focus on the case :

- $p = m$, $m = 2$ or $m = 3$
- $\mathcal{M} = \mathbb{S}^m$
- $\mathcal{N} = \mathbb{S}^2$.

for which $\pi_m(\mathcal{N}) = \mathbb{Z}$, so that we may study the asymptotic limit as $d \to +\infty$.

Energies of maps into spheres

We set for $p = 2$ and $p = 3$

$$\nu_p(d) = \inf\left\{E_p(w),\, w \in C^1(\mathbb{S}^p, \mathbb{S}^2)\ \deg_p(w) = d\right\}.$$

We consider the asymptotic properties as $|d|$ grows.

Theorem

We have:

- $\nu_2(d) = 8\pi|d|,\ \forall d \in \mathbb{Z}$
- $\nu_3(d) \propto |d|^{\frac{3}{4}}$ as $|d| \to +\infty$ *(Rivière, 98').*

Notice the difference of asymptotic growth!

The first case corresponds to degree theory, whereas the second relies on the Hopf invariant. We next provide some details on the proofs.

Lower bounds of the energy for maps $\mathbb{S}^2 \to \mathbb{S}^2$ through degree theory

Let $d \in \mathbb{Z}$ and consider a C^1 map $u : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\deg(u) = d$.. The integral formula for the degree yields

$$4\pi d = \int_{\mathbb{S}^2} u.u_x \times u_y \mathrm{d}x\, \mathrm{d}y$$

Invoking the inequality $u.u_x \times u_y \leq \frac{1}{2}(|u_x|^2 + |u_y|^2)$, we deduce

$$8\pi|d| \leq \int_{\mathbb{S}^2} |\nabla u|^2 \mathrm{d}x\, \mathrm{d}y$$

which yields the lower bound

$$\nu_2(d) \geq 8\pi|d|.$$

Upper bounds

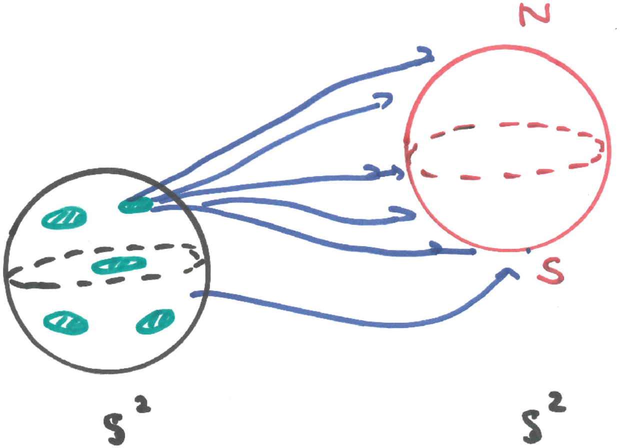
In order to show that

$$\nu_2(d) \leq 8\pi|d|,$$

we have to construct a sequence $(u_\varepsilon)_{\varepsilon>0}$ of maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ degree d maps such that

$$E_2(u_\varepsilon) \leq 8\pi|d| + \underset{\varepsilon\to 0}{O}(1)$$

We may prove this statement processing by gluing $|d|$ copies od degree 1 maps of energies close to 8π.


N
S
S^2
S^2

Energy lower-bounds for maps from $\mathbb{S}^3$ to $\mathbb{S}^2$

We use the integral formula for the Hopf invariant to deduce a lower bound for the energy.

Let $d \in \mathbb{Z}$ and consider a C^1 map $\mathrm{U}: \mathbb{S}^3 \to \mathbb{S}^2$ such that $\mathbf{H}(u) = d$.. The integral formula for the degree yields

$$16\pi^2|d| = \int_{\mathbb{S}^3} d^\star \Phi \wedge \mathrm{U}^\star(\omega_{\mathbb{S}^2}), \text{ with } \Delta_{\mathbb{S}^3}\Phi = \mathrm{U}^\star(\omega),$$

Since

$$\begin{cases} |\mathrm{U}^\star(\omega_{\mathbb{S}^2})| \leq C|\nabla \mathrm{U}|^2 \text{ and} \\ \qquad \|\nabla\Phi\|_{L^3} C \|\nabla \mathrm{U}\|^2_{L^3} \end{cases}$$

by elliptic estimates. We deduce by Hölder $6\pi^2|d| \leq \|\nabla \mathrm{U}\|^4_{L^3}$ so that

$$E_3(\mathrm{U}) \geq C d^{\frac{3}{4}}$$

so that $\nu_3(d) \geq C d^{\frac{3}{4}}$, as desired.

Upper-bounds for $\nu_3(d)$, Rivière 98'

It is more subtle and relies on the identity

$$H(\omega \circ \mathrm{U}) = (\deg \omega)^2 H(\mathrm{U}). \tag{2}$$

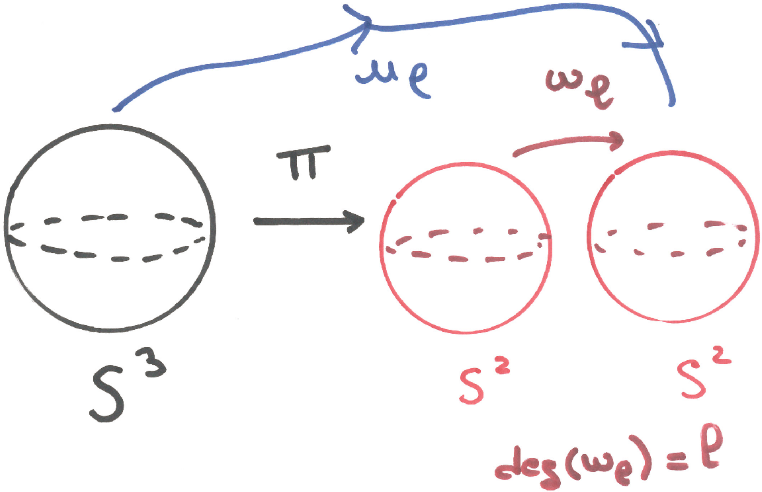
for $\mathrm{U} : \mathbb{S}^3 \to \mathbb{S}^2$ and $\omega : \mathbb{S}^2 \to \mathbb{S}^2$.

Notice the quadratic behavior with respect to ω

We apply this formula with $\mathrm{U} = \Pi$ the Hopf map, a map ω_ℓ of degree ℓ and set

$$\begin{cases} \mathrm{u}_\ell = \omega_\ell \circ \Pi, \text{ so that} \\ H(\mathrm{u}_\ell) = \ell^2. \end{cases}$$

We will construct ω_ℓ so that $\mathrm{E}_3(\mathrm{u}_\ell) \leq C|\ell|^{\frac{3}{2}} \leq C|\mathrm{H}(\mathrm{u}_\ell)|^{\frac{3}{4}}$.


u_ℓ
ω_ℓ
π
S^3
S^2
S^2
$\deg(\omega_\ell) = \ell$

One the construction of ω_ℓ

Given $\ell \in \mathbb{Z}$, we construct a smooth map $\omega_\ell : \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$\deg(\omega_\ell) = \ell \quad \text{and } |\nabla \omega_\ell|_{L^\infty(\mathbb{S}^2)} \leq C\sqrt{|\ell|},$$

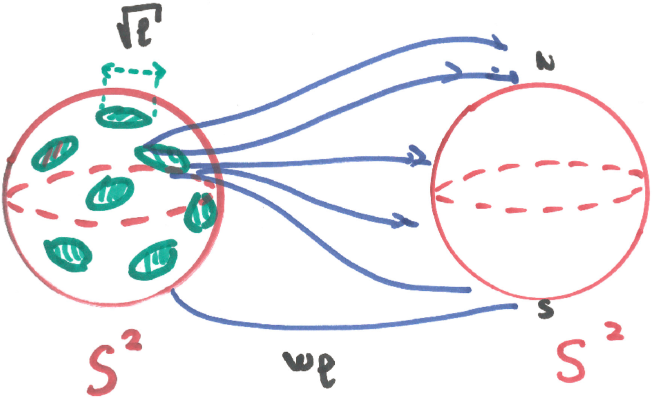
The idea is to glue together $|\ell|$ copies of de degree ± 1 maps scaled down to to cover disks of radii of order $\sqrt{|\ell|}$. This yields

$$\mathrm{E}_3(\mathrm{u}_\ell) \leq C|\ell|^{\frac{3}{2}} \leq C|\mathrm{H}(\mathrm{u}_\ell)|^{\frac{3}{4}},$$

yielding the estimate

$$\nu_d \leq |d|^{\frac{3}{4}}$$

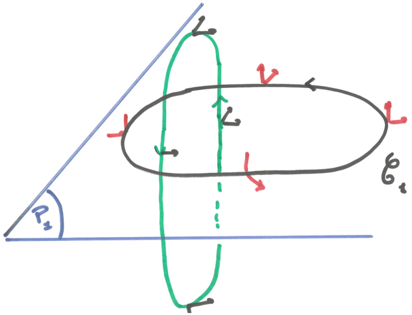
hence at least when the hopf invariant $d = \ell^2$ is a square.


$\sqrt{\varepsilon}$
N
S
S^2
w_ε
S^2

The spaghetton map

It corresponds to a slight modification of the previous construction, better adapted for later purposes. It relies on the Pontryagin construction.

As already seen, a way to create maps with non trivial topology is to consider two linked planar curves, with non twisted frames



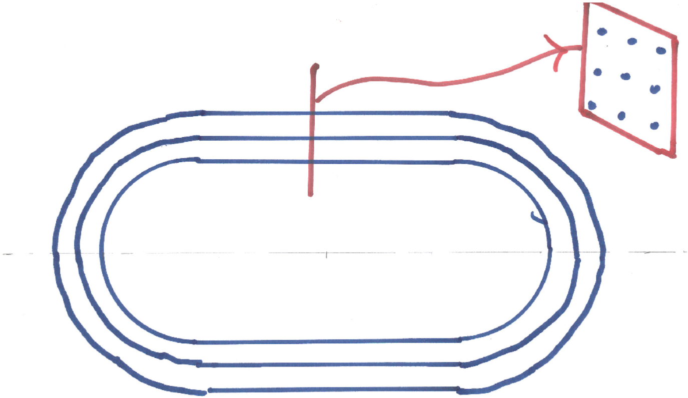
Hopf invariant of this map $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$ equals 2.

Sheaves of spaghetti

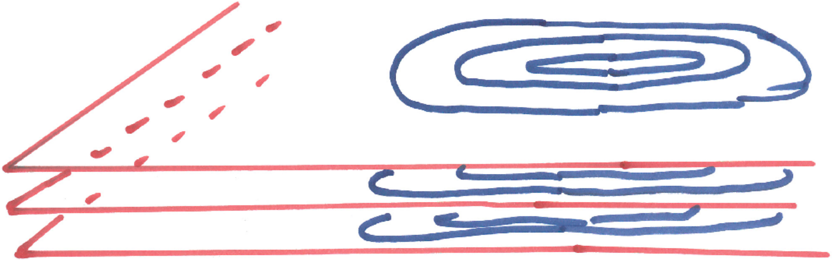
Instead of considering one single curve, we consider sheaves of ℓ^2 curves, which are planar, parallel, essentially of the same size :

- all curves are stadion shaped
- the mutual distance between two curves is of order $\sqrt{\ell^{-1}}$
- they lie in parallel planes of mutual distance of order $\sqrt{\ell^{-1}}$
- the interessection with an orthogonal plane passing through the middle od the stadions lies in a square of order 1.

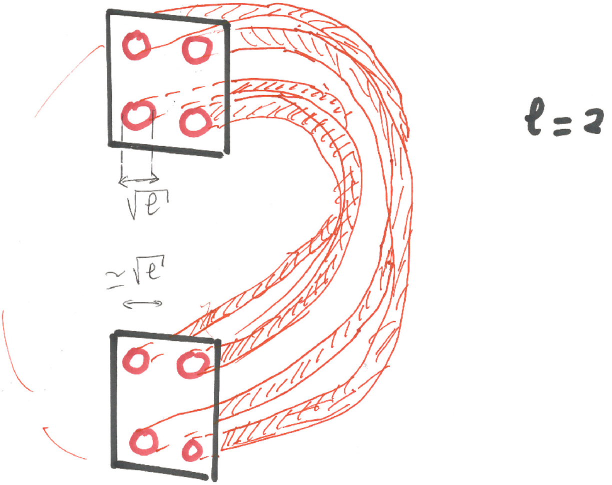
View from above of a sheave of 9 spaghetti



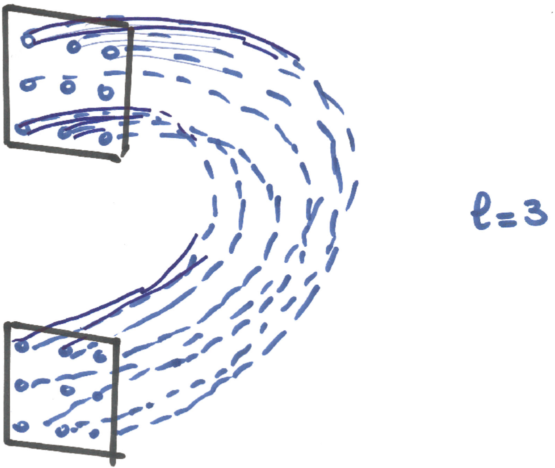
View from the side of a sheave of 9 spaghetti



A $\ell = 2$ sheave



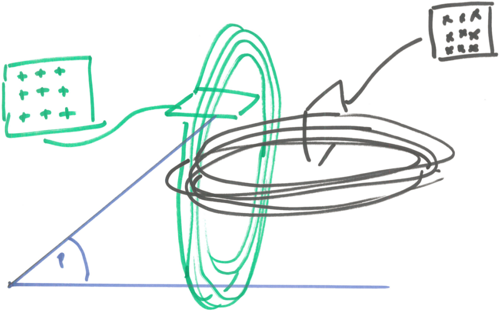
A $\ell = 3$ sheave



The k-Spaghetton map

Instead of considering one single sheave of spaghetti, we consider next two sheaves of spaghetti:

- for the same integer ℓ
- but lying in transversal direction, for instance parallel to the planes OxY and Oxz.
- the two sheaves are linked



We choose ϱ of order $\sqrt{\ell}$.

The Hopf invariant of the map $\mathfrak{S}_\ell \equiv \Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$, called the k spaghetton and denoted $\mathfrak{S}_\ell$, equals now $2\ell^4$.

Energy of the Spaghetton map

By appropriate choice of the thickness of the elementary spaghettis forming the sheaves of the spaghetton $\mathfrak{S}_\ell$ choosing $\varrho \propto \ell^{-1}$, one may show that

$$|\nabla \mathfrak{S}_\ell| \le C\ell$$

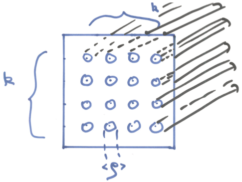
and equals 0 outside a region of measure of order ℓ^{-2}. This yields

$$\mathrm{E}_3(\mathfrak{S}_\ell) \simeq C\ell^3,$$

whereas the Hopf invariant is

$$\mathrm{H}(\mathfrak{S}_\ell) = 2\ell^4.$$

This yields again $\nu_d \le C d^{\frac{3}{4}}$, for $d = 2\ell^2$.



A gamma-convergence type results

Given a map $u \in W^{1,m}_S(\mathbb{B}^m, \mathcal{N})$, consider the set $\mathcal{V}(u)$ of all sequences $(v_n)_{n\in\mathbb{N}}$ such that

$$v_n \rightharpoonup u \text{ weakly in } W^{1,m}(\mathcal{M}, \mathcal{N}) \text{ and } [\![v_n]\!] = [\![v]\!],$$

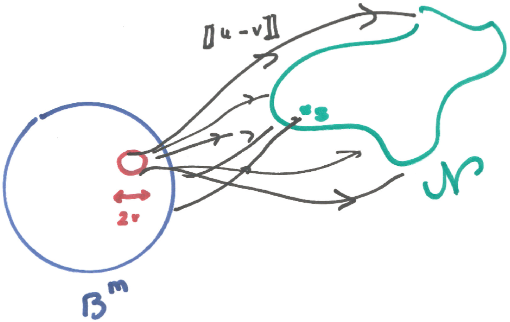
where $v \in W^{1,m}_S(\mathbb{B}^m, \mathcal{N})$ is given. the first obersvation is

Lemma

The set of sequences $\mathcal{V}(u)$ is not empty.

The proof amounts to construct explicitly a sequence $(v_n)_{n\in\mathcal{N}}$ which converges weakly to u. This can be done attaching "bubbles" to u, i. e maps in the homotopy class of $[\![v-u]\!]$, with radius r of order n^{-1} for instance.

Gluing bubbles



One may glue also at different points concentration of maps *alpha*$_1$ such that

$$\sum [\![\alpha_i]\!] = [\![u - v]\!].$$

Optimal sequences

If one seeks for an optimal sequence from the point of view of energy, then are led to consider the number

$$\begin{cases} \mu(u,v) = \inf\limits_{(v_n)_{n\in\mathcal{N}}\in\mathcal{V}} \left\{ \limsup\limits_{n\to+\infty} E(v_n) \right\} \text{ and} \\ \gamma(u,v) = \inf \left\{ \sum\limits_{\sum [\![\alpha_i]\!] = [\![u-v]\!]} \kappa_m(\alpha_i) \right\} \end{cases}$$

Theorem

We have

$$\mu(u,v) = E_m(u) + \gamma(u,v),$$

so that $\gamma(u,v)$ represents the minimal **the defect energy**.

As an example, if $m = p = 2$ and $\mathcal{N} = \mathbb{S}^2$, then we have

$$\mu(u,v) = E_2(u) + 8\pi|\deg u - \deg v|.$$

if $m = p = 3$ and $\mathcal{N} = \mathbb{S}^2$, then

$$\begin{cases} \mu(u,v) = E_3(u) + \inf\limits_{\sum d_i = \mathbf{H}(u) - \mathbf{H}(v)} \sum \nu_3(d_1) \\ \text{with } \ \nu(d_i) \propto |d_i|^{\frac{3}{4}}. \end{cases}$$

We next consider an optimal sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{V}(u)$such that

$$E_m(w_n) \underset{asn \to +\infty}{\to} \mu(u,v) = E_m(u) + \gamma(u,v)$$

This sequence has the following compactness properties

Compactness of optimal sequences

There exists a integer $k \in \mathbb{N}$, k points $a_1, \ldots a_k$ and homotopy classes $\alpha_1, \ldots, \alpha_k$ such that the following properties holds

- $w_n \to u$ strongly in $W^{1,m}(K)$ for any compact set $K \in \overline{\mathbb{B}^m} \cup \{a_i\}$.
- $|\nabla w_n|^p \to |\nabla u|^p + \sum_{i=1}^{k} \kappa_m(\alpha_i)\delta_{a_i}$ in the sense of measures
- $\sum_{i=1}^{k} \alpha_i = [\![u - v]\!]$.

This type of results can be used to show that infini are attained in some homotopy classes (Rivière, 98', $p = 3, \mathcal{N} = \mathbb{S}^2$).