Branched transportation and singularities of Sobolev maps between manifolds
Part III : Topological singularities

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Whereas we have focused so far on the case $p \geq m$, we investigate in this part properties of Sobolev maps in

$$W^{1,p}(\mathcal{M}, \mathcal{N})$$

In the case

$$1 \leq p < m = \dim \mathcal{M}$$

The discussion is very different due to the existence of topological singularities. An example of such a singularity is provided by the **Hedgehog**.

The hedgehog

The hedgehog map is given by, in dimension $m = 3$ by :

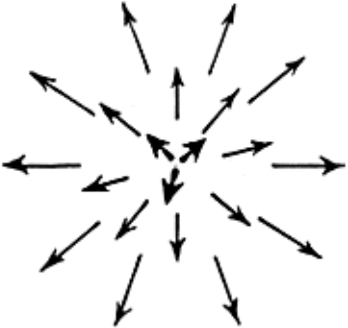
$$\mathcal{U}_{\mathrm{sing}}(x) = \frac{x}{|x|} \text{ for } x \in \mathbb{B}^3 \setminus \{0\}.$$

It is singular at the origin 0 but belongs to $W^{1,p}(\mathbb{B}^3, \mathbb{S}^2)$ iff $1 \le p < 3$ since

$$\mathrm{E}_p(\mathcal{U}_{\mathrm{sing}}) \equiv \int_{\mathbb{B}^3} |\nabla \mathcal{U}_{\mathrm{sing}}|^p = \int_0^1 r^{2-p} \left(\int_{\mathbb{S}^2} |\nabla(\mathrm{Id}_{\mathbb{S}^2})|^p \right) dr < +\infty.$$

Similarly, in dimension $m = 2$ belongs to $W^{1,p}(\mathbb{B}^2, \mathbb{S}^1)$ iff $1 \le p < 2$.

The hedgehog $\mathscr{U}_{\text{sing}}$ for $m = 3$

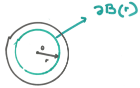


The hedgehog is **not** a strong limit of smooth maps for $p < m = 3$

Argue by contradiction: otherwise there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^3, \mathbb{S}^2)$ converging to $\mathscr{U}_{\mathrm{sing}}$. By mean-value a, $\exists 0 < r < 1$, such that

$$u_{n|_{\partial B(r)}} \underset{n\to\infty}{\to} \mathscr{U}_{\mathrm{sing}|_{\partial B(r)}} \text{ modulo a subsequence}$$

However $u_{n|_{\partial B(r)}}$ has trivial homotopy class whereas $\mathscr{U}_{\mathrm{sing}|_{\partial B(r)}}$ has not Brouwer theorem's, a contradiction in view of continuity of degree.



The same argument shows that the hedgehog is **NOT** the weak limit of smooth maps in $W^{1,p}(\mathbb{B}^3, \mathbb{S}^2)$ $2 < p < 3$.

The same kind of counter-examples show that :

- No way to define homotopy classes for $p < 3 \leq m$ in $C^1(\mathbb{S}^m, \mathbb{S}^2)$.
- No lifting property for $W^{1,p}(\mathbb{B}^2, \mathbb{S}^1)$ maps, for $1 \leq p < 2$

The general case $1 \leq p < m = \dim \mathcal{M}$

As seen before on an example, singularities play an important role in the description of properties of maps. This raises a central question:
Of course, one may first raise the question:

Question 0 : Is $C^\infty(\mathcal{M}, \mathcal{N})$ *strongly dense* in $W^{1,p}(\mathcal{M}, \mathcal{N})$?
which has now a rather satisfactory set of answers.

One may similar raise the same question on the level of weak density

Question 1 : Is $C^\infty(\mathcal{M}, \mathcal{N})$ *sequentially weakly dense* in $W^{1,p}(\mathcal{M}, \mathcal{N})$?

Notice that

$$\textbf{YES to question 0} \Longrightarrow \textbf{YES to Question1}.$$

Strong density

Answer is **NO to strong density** if $1 \le p < m$, $\pi_{[p]}(\mathcal{N}) \neq \{0\}$ *due to the existence of topological singularities*

$[p]$ = largest integer less or equal to p.

The counter-examples have the same flavor as for the standard Hedgehog.

For instance, if $m-1 \le p < m$ so that $[p] = m-1$, and the assumption is $\pi_{m-1}(\mathcal{N}) \neq \{0\}$, then a map which cannot be approximated by smooth maps for $\mathcal{N} = \mathbb{B}^m$ can be constructed as follows :

$$\mathcal{U}_{\text{sing}} = \varphi\left(\frac{x}{|x|}\right)$$

where $\varphi : \mathbb{S}^{m-1} \to \mathcal{N}$ is a map in a non -trivial homotopy class.

Singularities and weak density

We have

Theorem

If $1 \leq p \leq m$ is **not an integer** *and $\pi_{[p]}(\mathcal{N}) \neq \{0\}$ then the answer to the main question is NO, that is smooth maps are not sequentially weakly dense in $W^{1,p}(\mathcal{M}, \mathcal{N})$.*

We have already illustrated the argument on a simple example, the case $\mathcal{M} = \mathbb{B}^3$, $\mathcal{N} = \mathbb{S}^2$, the argument carrying over to the general case using the same argument as in the previous slide.

Necessary condition for strong density

The condition $\pi_{[p]}(\mathcal{N}) = \{0\}$ turns out to be also necessary if $1 \leq p < m$, $\pi_{[p]}(\mathcal{N}) = \{0\}$, $\mathcal{M}$ has a simple topology
[B 91, counterexamples for complicated topologies of $\mathcal{M}$ in Han-Lin 01...]

Proofs requires some sophisticated constructions of approximating maps

Maps with finitely many singularities

When approximability by smooth maps does not hold, a natural question is to seek for sets of maps with "maximal possible regularity" which are dense in $W^{1,p}(\mathcal{M},\mathcal{N})$.
In this direction, consider, for $m-1 \leq p < m$ the set of maps which are smooth, except possibly at a finite number of points, that is

$$\mathcal{R}(\mathcal{M},\mathcal{N}) = \{u \in W^{1,p}(\mathcal{M},\mathcal{N}), \text{ s.t } u \in C^\infty(\mathbb{B}^m \setminus \{A\}) \text{ for a finite set } A\}.$$

Theorem

if $m-1 \leq p < m$, then $\mathcal{R}(\mathcal{M},\mathcal{N})$ is **dense** *in $W^{1,p}(\mathcal{M},\mathcal{N})$*

More generally, maps with singular set of codimension $m-[p]-1$ are dense in $W^{1,p}(\mathcal{M},\mathcal{N})$. This sets can be used for various purpose, as smooth maps in classical results analysis.

Weak density: First remarks

We next focus throughout on sequentially weak density.

The first observation is that if strong approximation holds, this is also the case at the level of sequentially weak approximation :

Hence we may assume throughout that

$$\pi_{[p]}(\mathscr{N}) \neq \{0\}.$$

As we have already seen, the obstruction to strong density related to the presence of singularities remains for weak density, provided however p **is NOT an integer**.

Sequentially weak density

The only open case is hence given by the following: **Open case:** $1 \leq p < m$, $\pi_p(\mathcal{N}) \neq \{0\}$, and p **is an integer**.

The answer depends crucially on further properties of $\mathcal{N}$.
I will discuss two cases, which have been handled so far:
The first is $\mathcal{N} = \mathbb{S}^p$ so that $\pi_p(\mathcal{N}) = \mathbb{Z}$ is related to standard **degree theory**:

Theorem (B-Zheng 88, B 91)

Let p be an integer. Then given any manifold $\mathcal{M}$, $C^\infty(\mathcal{M}, \mathbb{S}^p)$ is sequentially weakly dense in $W^{1,p}(\mathcal{M}, \mathbb{S}^p)$.

In constrast, we will show in the case $p = 3$ and $\mathcal{N} = \mathbb{S}^2$:

Theorem (B 14)

Given any manifold $\mathcal{M}$ of dimension larger then 4*, $C^\infty(\mathcal{M}, \mathbb{S}^2)$ is* **not sequentially weakly dense** *in $W^{1,3}(\mathcal{M}, \mathbb{S}^2)$.*

Strongly related to properties of the Hopf Fibration.

Perhaps more surprising at first sight are the connection with **optimal transportation**.

Remark Notice that the question remains unsolved in the general case.

Weak density in $W^{1,2}(\mathscr{M}, \mathbb{S}^2)$

Let $u \in W^{1,2}(\mathscr{M}, \mathbb{S}^2)$. A simple argument shows that one may approximate u weakly in $W^{1,2}(\mathscr{M}, S^2)$ by smooth maps. For $A \in \mathbb{S}^2$ and $0 < \varepsilon \le 1$ consider the set $\mathbb{C}(A, \varepsilon)$ diffeomorphic to a disk

$$\mathbb{C}(A,\varepsilon) = \mathbb{S}^2 \setminus \mathbb{B}^3(A,\varepsilon)$$

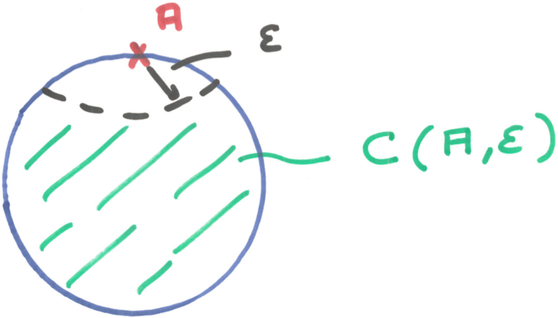
and a map $\Phi(A,\varepsilon) : \mathbb{S}^2 \to \mathbb{C}(A,\varepsilon)$ such that

$$\begin{cases} \Phi(A,\varepsilon)(x) = x \text{ for } x \in C(A,\varepsilon) \\ |\nabla\Phi(A,\varepsilon)| \le C\varepsilon^{-1} \end{cases}$$

We first approximate u weakly by maps of the form

$$w_n = \Phi(A_n, \varepsilon_n) \circ u \in W^{1,p}(\mathscr{M}, \mathbb{C}(A,\varepsilon)),$$

with $\varepsilon_n \to 0$ and A_n suitably choosen. We then approximate the later strongly by smooth maps: this is easy since their are with values into a chart, namely $C(A,\varepsilon)$.



The point A_n is choosen **by averaging** so that in particular

$$\int_{u(x)\in\mathbb{C}(A_n,\varepsilon_n)} |\nabla u|^2 \leq C\varepsilon_n^2.$$

It follows that

$$\begin{aligned} \int |\nabla w_n|^2 &\le \int_{u(x)\notin \mathbb{C}(A_n,\varepsilon_n)} |\nabla u|^2 + C\varepsilon^{-2} \int_{u(x)\in \mathbb{C}(A_n,\varepsilon_n)} |\nabla u|^2 \\ &\le C \int |\nabla u|^2, \end{aligned}$$

Hence the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in H^1 and soft argument show that passing possibly to a further subsequence it converges to u

Singularities of maps in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$

We turn with a rather different point of view to the case $\mathcal{M} = \mathbb{B}^3$, $p = 2$ and maps from in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$.
We stress here **two main facts**:

- $C^\infty(\mathbb{B}^3, \mathbb{S}^2)$ is **not dense** in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$
- $C^\infty(\mathbb{B}^3, \mathbb{S}^2)$ is **sequentially weakly dense** in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$

The hedgehog is a weak limit of smooth maps

Gluing small bubbles, we construct first $(\varphi_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ s. t.

- $\deg(\varphi_n) = 0$
- $\varphi_n(x) = x$ for any $|x - S| \geq n^{-1}$
- $\mathrm{E}_2(\varphi_n) \leq 2\mathrm{E}_2(\mathrm{Id}_{\mathbb{S}^2}) + \frac{1}{n} = 8\pi + \frac{1}{n}$.

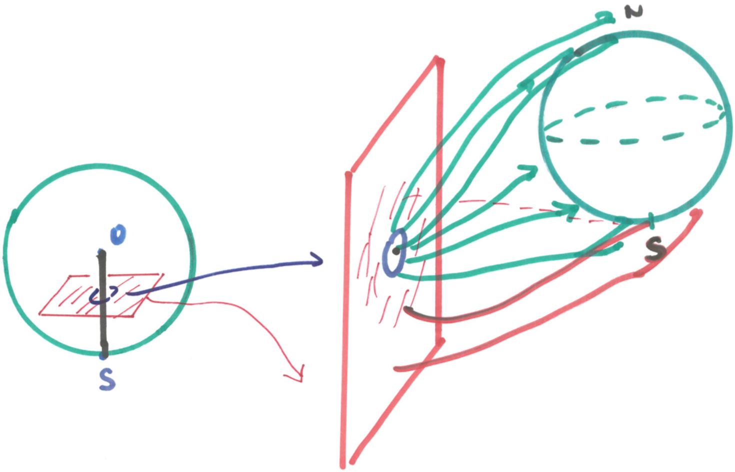
We set

$$U_n(x) = \varphi_n\left(\frac{x}{|x|}\right) \text{ for } \frac{1}{n} \leq |x| \leq 1.$$

and extend U_n inside $\mathbb{B}(\frac{1}{m})$ is a smooth way with small energy. The energy of the sequence $(U_n)_{n\in\mathbb{N}}$ hence concentrates on the segment $[0,S]$:

$$|\nabla U_n|2 \rightharpoonup |\nabla \mathscr{U}_{\text{sing}}|^2 + 4\pi\mathscr{H}^1 \llcorner [0,S] \text{ in the sense of measures on } \mathbb{B}^3$$

yielding the desired weak approximation.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Using maps with a finite number of singularities

As seen the set of maps with a finite number of isolated singularities

$$\mathscr{R}(\mathbb{B}^3, \mathbb{S}^2) = \{u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2), \text{ s.t } u \in C^\infty(\mathbb{B}^m \setminus \{A\}) \text{ for a finite set } A\}.$$

is **dense** in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ [B-Zheng, 88]. We may assume moreover that all singularities have degree$+1$ or -1.

A map $v \in \mathscr{R}$ can be weakly approximated by smooth maps as for $\mathscr{U}_{\text{sing}}$, using concentration of bubbles along lines connecting the singularity to the boundary, or possibly to other singularities with opposite charges.

Considering segments $\mathscr{L}_i$ joining the singularities of opposite charges or to the boundary, we obtain a sequence of smooth maps $(\varphi_n)_{n\in\mathbb{N}}$ such that

$$|\nabla \varphi_n|^2 \rightharpoonup |\nabla v|^2 + \mu_* \text{ as } n \to +\infty \quad \text{where } \mu_* = 8\pi \mathscr{H}^1 \llcorner \left(\bigcup_{i=1}^{r} \mathscr{L}_i \right),$$

The measure μ_* represents **the defect energy measure for the convergence**. The mass ϵ_* of μ_* represents **the defect energy**.

Defect measure and energy, minimal connections

Weak approximability by smooth maps turns hence into bounds for ϵ_*:

$$\lim_{n\to\infty} \mathrm{E}_2(\varphi_n) = \mathrm{E}_2(v) + \epsilon_* \text{ where } \epsilon_* = |\mu_*| = \nu_{\mathcal{N}}(1)\left(\sum_{i=1}^{r} \mathcal{H}^1(\mathcal{L}_i)\right),$$

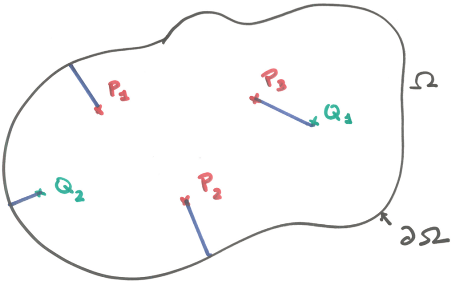
leading to the notion of minimal connection introduced by Brezis, Coron and Lieb. Let $\{P_i\}_{i\in J}$ denote the set of singularities of degree+1, $\{Q_i\}_{i\in J}$ of charge -1, adding possibly some fictitious singularities on the boundary so that the total charge is zero. The lenght of a minimal connection writes

$$\mathrm{L}(\{P_i\},\{Q_i\}) = \inf\left\{\sum_{i\in J} |P_i - Q_{\sigma(i)}|, \text{ for } \sigma \in \mathfrak{S}\right\},$$

where $\mathfrak{S}$ the set of perturbations of J. Going back to (23) we obtain

$$\epsilon_* = \epsilon_*(v) = |\mathbb{S}^2|\mathrm{L}(v) \text{ where } \mathrm{L}(v) \equiv \mathrm{L}(\{P_i\},\{Q_i\}) \text{ since } \nu_{\mathcal{N}}(1) = |\mathbb{S}^2|.$$

The notion of lenght of a minimal connection is closely related, up to the presence of charges of opposite sign, to the functional $\mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A, \partial\Omega)$, for $\alpha = 1$.



This case corresponds to optimal transportation.

An important observation by Brezis, Coron and Lieb

They observed that L can be related to the energy of the map as

$$\mathrm{E}_2(v) \geq 2|\mathbb{S}^2|\mathrm{L}(v) = |\mathbb{S}^2|\mathrm{L}(\{P_i\},\{Q_i\}),$$

so that the defect energy $\epsilon_*(v)$ is bounded by the Dirichlet energy

$$\epsilon_*(v) \leq \mathrm{E}_2(v).$$

This fact, combined with the density of $\mathscr{R}$, allows to show that any map in $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ is the weak limit of smooth maps (B, 91').

Remark

There is a beautiful proof of the Brezis-Coron-Lieb result due to Almgreen-Browder-Lieb relying on the coarea formula. We have for any map in $\mathscr{R}$

$$\int_{\mathbb{S}^2} \mathscr{H}^1\left(u^{-1}(\theta)\right) \mathrm{d}\theta = \int_\Omega |Ju| \mathrm{d}x \leq \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x$$

where:

- $u^{-1}(\theta)$ denotes the counter-image of any arbitrary point θ on $\mathbb{S}^2$
- $\mathscr{H}^1(u^{-1}(\theta))$ its length
- $|Ju|$ denotes the jacobian of the map restricted to the orthogonal to the null-space

It can be shown that $\mathscr{H}^1(u^{-1}(\theta)$ is always larger than the minimal connection between the singularities, leading to the proof.

We will next to the case $p = 3$, $\mathcal{M} = \mathbb{B}^4$ and show that, in that case there exist maps in

$$W_{\mathrm{S}}^{1,3}(\mathbb{B}^4, \mathbb{S}^2) = \{u \in W^{1,3}(\mathbb{B}^4, \mathbb{S}^2), u(x) = \mathrm{S} \text{ for } \mathrm{x} \in \partial \mathrm{B}^4\}$$

that are **NOT** weak limits of smooth maps.

An already mentioned, the main result we wish to discuss is the following

Proposition

There exists a map $\mathscr{U}$ is $W^{1,3}_{\mathrm{S}}(\mathbb{B}^4, \mathbb{S}^2)$ which is not the weak limit of smooth maps between $\mathbb{B}^4$ and $\mathbb{S}^2$.

This property holds though the spaces $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$ and $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ have many common properties:

- We have $\pi_3(\mathbb{S}^2) = \mathbb{Z}$ (compare with $\pi_2(\mathbb{S}^2) = \mathbb{Z}$)
- Homotopy classes of continuous maps from $\mathbb{S}^3$ to $\mathbb{S}^2$ are labelled by an integer denoted below $\deg_3$, called the **Hopf invariant** (compare with degree theory)
- The set of maps with a finite number of isolated singularities

$$\mathscr{R}(\mathbb{B}^3, \mathbb{S}^2) = \{u \in W^{1,3}(\mathbb{B}^4, \mathbb{S}^2), \text{ s.t } u \in C^\infty(\mathbb{B}^m \setminus \{A\}) \text{ for a finite set } A\}.$$

of Hopf number ± 1 is **dense** in $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$

The main different between $W^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$ and $W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$

It occurs on the level of energy estimates. Set as before for $p = 2$ and $p = 3$

$$\nu_p(d) = \inf \left\{ E_p(w),\ w \in C^1(\mathbb{S}^p, \mathbb{S}^2)\ \deg_p(w) = d \right\}.$$

Recall:

- $\nu_2(d) = 8\pi|d|,\ \forall d \in \mathbb{Z}$ (invoking integral formulation of degree theory)
- $\nu_3(d) \propto |d|^{\frac{3}{4}}$ as $|d| \to +\infty$. (Rivière, 98').

$$\Longrightarrow$$

High multiplicity is favored when concentrating bubbles along lines.

Optimal transport has to be replaced by **branched transportation** with

$$\alpha = \alpha_4 = \frac{3}{4} \text{ critical exponent in dimension } m = 4.$$

The following is the main ingredient in the construction of the map $\mathscr{U}$:

Lemma

Given any $k \in \mathbb{N}^*$, there exists a map $\mathfrak{v}_k \in \mathscr{R}_{\mathrm{S}}(\mathbb{B}^4, \mathbb{S}^2)$ such that

$$\begin{cases} \mathrm{E}_3(\mathfrak{v}_k) \le \mathrm{C}_1 k^3, \mathrm{C}_1 > 0 \\ \mathrm{L}_{\mathrm{branch}}(\mathfrak{v}_k) \ge \mathrm{C}_2 \log(k) k^3, \ rC_2 > 0 \end{cases}$$

The functional $\mathrm{L}_{\mathrm{branch}}$ refers to:

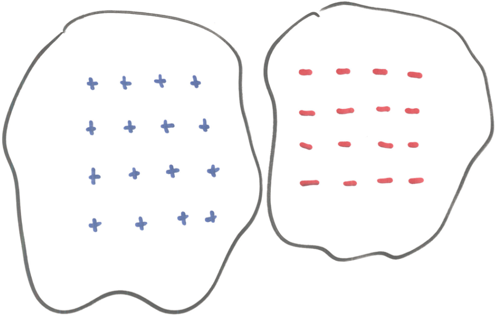
- a branched transportation with exponent $\frac{3}{4}$ connecting singularities of opposite signs or to the boundary, analogous to the length L of a minimal connection in dimension 3.
- It also yields the minimal defect energy for the weak approximation by smooth maps (Hardt-Rivière 03').

$$\text{defect energy} \simeq \mathrm{L}_{\mathrm{branch}}(\mathfrak{v}_k) \ge C(\log k)\,\mathrm{E}_3(\mathfrak{v}_k),$$

(compare with the result of Brezis, Coron and Lieb)

comments

The function $\mathfrak{v}_k$ of the Lemma has k^4 singularities of charge $+1$, as well as k^4 singularities of charge -1. These $+1$ are located on a uniform grid, far from the negative charges.



Some ideas in the proof of the Key Lemma

The central point is to deform the k-Spaghetton map $\mathfrak{S}_k$ to a constant map. Since, for $k \in \mathbb{N}^*$, $\mathbf{H}(\mathfrak{S}_k) \neq 0$ it is not possible to do it within continuous maps.This becomes possible work instead in $W^{1,3}$. In short:

In the continuous class, the two sheafs are not allowed to cross. In constrast, in the Sobolev class $W^{1,3}$ they are!

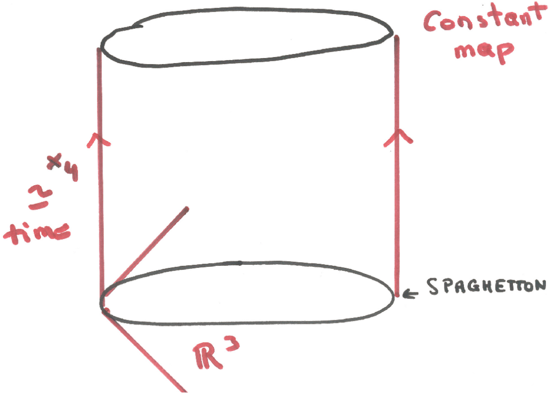
We consider the strip Λ of $\mathbb{R}^4$ defined by

$$\Lambda = \mathbb{R}^3 \times [0,30] = \{(x', x_4),\ x' \in \mathbb{R}^3,\ 0 \le x_4 \le 30\},$$

The set Y_k of maps $w : \Lambda \to \mathbb{S}^2$ such that:

$$\begin{cases} \mathrm{E}_3(w,\Lambda) \equiv \displaystyle\int_\Lambda |\nabla w|^3 < \infty \\ w(x',0) = \mathfrak{S}_k(x',0) \text{ and } w(x',30) = S \ \text{ for almost every } x' \in \mathbb{R}^3 \\ w(x',s) = S \text{ for every } x' \in \mathbb{R}^3 \text{ such that } |x'| \ge 30 \text{ and } 0 \le s \le 30, \end{cases}$$

is hence not empty!


Constant
map
x_4
=
time
SPAGHETTON
$\mathbb{R}^3$

Lemma

There exists a map $\mathfrak{C}_k$ in Y_k such that $\mathfrak{C}_k$ has exactly k^4 topological singularities of charge +2 and such that

$$\mathrm{E}_3(\mathfrak{C}_k) \le 10\, \mathrm{C}_{\mathrm{spg}}\, k^3. \tag{1}$$

If Υ^k denotes the set of singularities of $\mathfrak{C}_k$, then

$$A_0^h + k^{-1} \left(\bigcup_{i,j=1}^{k} \bigcup_{q,r=1}^{[\frac{k}{2}]} \{(i,j,q,2r)\} \right) \subset \Upsilon^k \subset A_0^h + [0,1]^3 \times [-2,2], \tag{2}$$

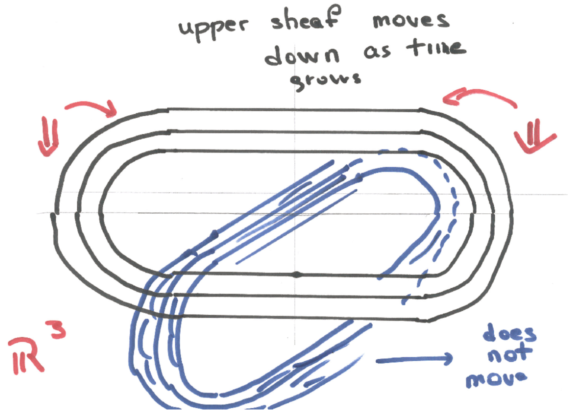
where $A_0^h = (0, -1-h, -\frac{1}{8}h, 4)$

Recall that

$$\mathrm{E}_3(\mathfrak{S}_k) \le Ck^3,$$

which is consistent with the estimate for $\mathfrak{C}_k$ (also$\propto k^3$).

The heuristic idea of the proof of Proposition 6 is to consider the x_4 variable as a time variable. Our deformation of the spaghetton then consists in moving parts of the fibers onto the other parts so that they are ultimately unlinked. However, in order to do so, crossings are inevitable, each of them yielding a singularity of $\mathfrak{C}_k$.


upper sheaf moves down as time grows
$\mathbb{R}^3$
does not move

End of the proof of the key Lemma

the maps v_k in the Key Lemma is deduce from $\mathfrak{C}_k$ using a few elementary transformations :

- symetries
- Dilations
- change of frames, etc..

and is completely elementary

The key Lemma yields counter-examples to weak density

The map $\mathscr{U}$ described in the main theorem above is obtained:

- pasting a infinite countable number of copies of scaled and translated versions of the maps $\mathfrak{v}_k$ for suitable choices of the integer k and the scaling factors.
- This gluing is performed in such a way that the energies sum up to provide a finite total energy whereas the values for the respective functional $\mathrm{L}_{\text{branch}}$ do not: this is made possible since the two quantities behave differently as k grows.
- The conclusion then immediately follows from the convergence by Hardt and Rivière.