Spaces of Flat and Normal Chains and Cochains

Robert Hardt
(Rice University)

Lyon Winter School, “Nonlinear function spaces in mathematics and physical sciences”

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Coauthors

Thierry De Pauw (Paris VII) -H.,
Rectifiable and Flat G Chains in a Metric Space Amer.J.Math. 2011

DePauw, -H.
Some basic theorems on flat G chains, J. Math. Anal. Appl. 2014.

De Pauw, -H., Washek Pfeffer (UC Davis, Emeritus)
Homology of Normal Chains and Cohomology of Charges To appear in Memoirs AMS.

Outline

Lecture I. Rectifiable and Flat Chains and the Plateau Problem

Lecture II. Normal Chains, Cochains, and Charges

Lecture III. The Linear Isoperimetric Property

A Vague Problem from Algebraic Topology

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* For M triangulated, use simplicial theory.
* For M a smooth manifold and for real coefficients, use differential forms and De Rham theory.
* For M semi-algebraic, use semi-algebraic chains, etc.

Minimizers

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Example 2. B is three (similarly-oriented) semi-circles bounding A which is three half-disks. Here $\partial B = 0$ and $\partial A = B$ as $\mathbb{Z}/3\mathbb{Z}$ chains.

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2002 Jerrard, **2003** H.-DePauw, **2005** T. Adams, **2007** S. Wenger, **2007** U. Lang, **2009** Ambrosio-Wenger, **2009** Ambrosio-Katz, **2009** M. Snipes, **2010** C. Riedweg, **2011** Wenger, **2013** Rajala-Wenger, **2015** Camille-Rajala-Wenger.

Rectifiable Chains

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What are rectifiable chains?

Rectifiable Sets

A subset M of a metric space X is $\mathcal{H}^m$ *rectifiable* if $\mathcal{H}^m(M \setminus f(E)) = 0$ for some Lebesgue measurable $E \subset \mathbb{R}^m$ and Lipschitz $f : E \to M$.
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Parameterization Theorem. *There exist disjoint compact $A_i \subset \mathbb{R}^m$ and an injective map $\alpha : A = \cup_{i=1}^{\infty} A_i \to M$ such that $\mathcal{H}^m[M \setminus \alpha(A)] = 0$, $\mathrm{Lip}\,\alpha \le 1 + \delta$, and $\mathrm{Lip}(\alpha \restriction A_i)^{-1} \le 2\sqrt{m}$.*

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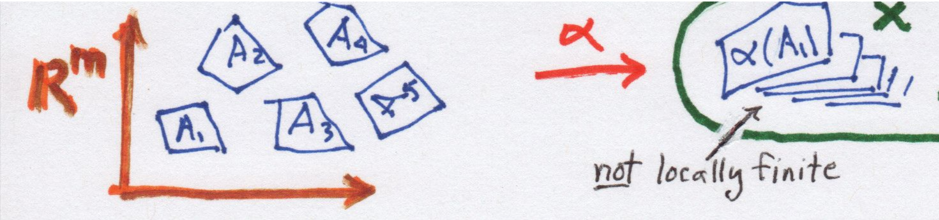

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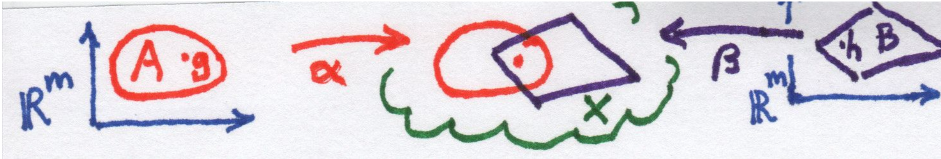

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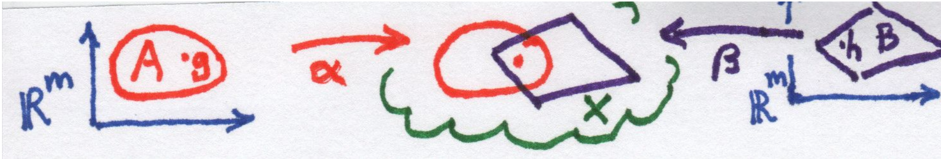

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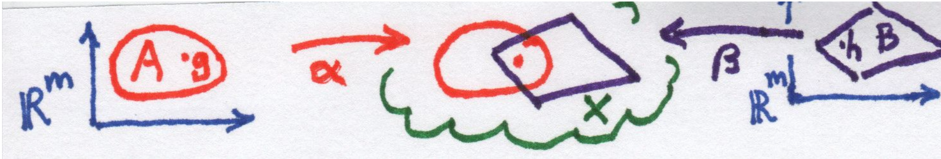

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To take advantage of some hidden linear structure for rectifiable objects in a metric space we will use:

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This argument also gives isometric embeddings in $\mathrm{Lip}_b(X) \subset \mathcal{C}_b(X)$ with the *sup* norm.

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Suppose $Y = \ell^\infty(D)$ contains X as before.

A *polyhedral G chain* in Y is simply a finite sum $P = \sum_{i=1}^{I} [\![\gamma_i, \Delta_i, g_i]\!]$ where $\gamma_i : \mathbb{R}^m \to Y$ is affine, Δ_i is an m simplex, and g_i is *constant* on Δ_i.

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As the Koch snowflake in the plane shows, the boundary of a rectifiable chain is not expected to be rectifiable in general. So defining it requires completion of Lipschitz chains with respect to a weaker norm.

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(Federer's "Lipschitz" Sard Theorem)

Push-forward and Slicing

Suppose $T \in \mathcal{R}_m(X;G)$.

For an injective Lipschitz map ϕ from X to another metric space Y we can use a representation $T = [\![\alpha, A, g]\!]$ to well-define the *push-forward* $\phi_\# T = [\![\alpha \circ \phi, A, g]\!] \in \mathcal{R}_m(Y;G)$. For a general Lipschitz $\phi : X \to Y$, one can make a suitable partition of the domain and range as in the proof of the parameterization theorem.

For a Lipschitz $f : X \to \mathbb{R}^n$ with $n \leq m$ and a.e. $y \in \mathbb{R}^m$, one may define define the *slice* of T in $f^{-1}\{y\}$, $\langle T, f, y\rangle \in \mathcal{R}_{m-n}(X;G)$ by first treating the case $T = [\![\mathrm{id}, A, g]\!] \in \mathcal{R}_m(\mathbb{R}^m;G)$. Here for a.e. $y \in \mathbb{R}^n$, one has for $\mathcal{H}^{m-n}$ a.e. $x \in A \cap f^{-1}\{y\}$ that f is differentiable at x with rank n. One obtains a parameterization for $A \cap f^{-1}\{y\}$ near x.
(Federer's "Lipschitz" Sard Theorem)

For the general case, let

$$\langle [\![\alpha, A, g]\!], f, y\rangle = \alpha_\# \langle [\![\mathrm{id}, A, g]\!], f \circ \alpha, y\rangle \ .$$

Slicing via Sublevel Sets

In case $n = 1$, and $\mathbb{M}(T) + \mathbb{M}(\partial T) < \infty$, we have, for a.e. $r \in \mathbb{R}$, the handy boundary restriction formula

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Here we may, for a.e. r, replace the set $\{f < r\}$ by $\{f \leq r\}$ in either or both occurrences because the set $f^{-1}\{r\}$ has both μ_T and $\mu_{\partial T}$ measure zero, except for at most countably many r.

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In case $n > 1$, we may write $f = (f_1, f_2, \cdots, f_n)$, and we have for a.e. $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ the formula

$$\langle T, f, y \rangle \;=\; \langle \cdots \langle T, f_1, y_1 \rangle, \cdots, f_n, y_n \rangle \; ,$$

expressing the $\mathbb{R}^n$ slice as repeated $\mathbb{R}$ slices.

Flat Norm and Flat Chains

Note that in the space $\mathbb{R}$ the points $1/i$ approach the point 0, but the corresponding 0 dimensional chains $[\![1/i]\!]$ do not approach $[\![0]\!]$ *in mass norm* because $\mathbb{M}([\![1/i]\!] - [\![0]\!]) = 2$.

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Whitney defined the *flat norm*, which we adapt. For a Lipschitz chain $T \in \mathcal{L}_m(Y; G)$, let

$$\mathcal{F}(T) \;=\; \inf\{\mathbb{M}(S) + \mathbb{M}(T - \partial S) \;:\; S \in \mathcal{L}_{m+1}(Y, G)\} \,.$$

Then the flat norm $\mathcal{F}([\![1/i]\!] - [\![0]\!]) \;\le\; 1/i \;\to\; 0$ because $[\![1/i]\!] - [\![0]\!] = \partial[\![0, 1/i]\!]$.

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Then $\mathcal{F}$ is a norm on $\mathcal{L}_m(Y; G)$, and we define the group of *flat chains* $\mathcal{F}_m(Y; G)$ as the $\mathcal{F}$ completion of $\mathcal{L}_m(Y; G)$ (or of $\mathcal{P}_m(Y; G)$ [De Pauw]).

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Since $\mathcal{F} \le \mathbb{M}$, a rectifiable chain $T \in \mathcal{R}_m(Y; G)$ is flat and so now has a well-defined boundary $\partial T \in \mathcal{F}_{m-1}(Y; G)$.

Slicing Flat Chains

Using the integral mass slice estimate $\int_{\mathbb{R}^n} \mathbb{M}\langle T, f, y\rangle dy \leq c(\mathrm{Lip}\, f)^n \mathbb{M}(T)$, which leads to a corresponding integral *flat norm* slice estimate, we readily extend slicing to flat chains.

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Note that this implies that, for a.e. $-\infty < r < s < +\infty$,

$$\langle T, f, s\rangle - \langle T, f, r\rangle = \partial(T \llcorner f^{-1}[r, s]) - (\partial T) \llcorner f^{-1}[r, s]\ ,$$

which give the flat norm estimate

$$\mathcal{F}(\langle T, f, s\rangle - \langle T, f, r\rangle) \le \mathbb{M}(T \llcorner f^{-1}[r, s]) + \mathbb{M}((\partial T) \llcorner f^{-1}[r, s]\ )\ .$$

A Total Variation Estimate for the Slice

For a.e. finite sequence $-\infty < r_0 < r_1 < \cdots < r_I < \infty$, we deduce

$$\sum_{i=1}^{I} \mathcal{F}(\langle T, f, r_{i+1}\rangle - \langle T, f, r_i\rangle) \leq c\mathbb{N}(T) \ .$$

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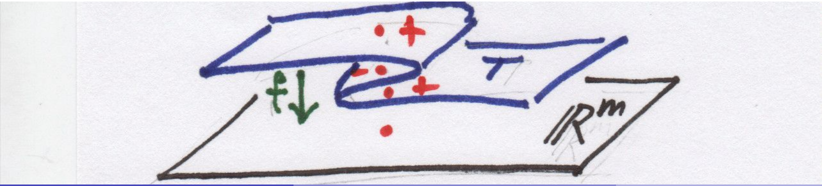
For any $T \in \mathcal{F}_m(X)$ *with* $\mathbb{N}(T) = \mathbb{M}(T) + \mathbb{M}(\partial T) < \infty$*, the slice function*

$$\langle T, f, \cdot\rangle \ \in \ \mathrm{MBV}\,(\mathbb{R}^m, \mathcal{F}_0(X; G))$$

with total variation bounded by $C\mathbb{N}(T)$*.*



Lower Semicontinuity ?

Theorem. *If* $T_i, T \in \mathcal{L}_0(X; G)$ *and* $\mathcal{F}(T_i - T) \to 0$*, then*

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Theorem. *There is an* $\mathcal{F}$ *lower semicontinuous norm* $\hat{\mathbb{M}}$ *on* $\mathcal{L}_m(X; G)$ *with* $m^{-m}\hat{\mathbb{M}} \ \leq \ \mathbb{M} \ \leq \ \hat{\mathbb{M}}$.

Definition of $\hat{\mathbb{M}}$

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In case $X = \mathbb{R}^N$ and $(G, \| \|) = (\mathbb{R}, | |)$, any $T \in \mathcal{R}_m(X; G)$ defines a *rectifiable current*, any smooth $f : \mathbb{R}^N \to \mathbb{R}^m$ gives the simple m form $df_1 \wedge \cdots \wedge df_m$, and we have the formula

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We get the "supremum" measure

$$\hat{\mu}(A) \ = \ \sup \left\{ \sum_{i=1}^{I} \mu_{U_i,f}(A) \ : \ U_i \text{ are disjoint open in } X, \ f \in \mathrm{Lip}_1(X, \mathbb{R}^m) \right\}$$

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Remark. $\mathbb{M}(T) = \hat{\mathbb{M}}(T)$ *if $m = 0$, $m = 1$, or X isometrically embeds in a Hilbert space.*

Finally defining, for $T \in \mathcal{F}_m(X; G)$,

$$\hat{\mathbb{M}}(T) = \liminf_{\delta \downarrow 0} \{\hat{\mathbb{M}}(L) \ : \ L \in \mathcal{L}_m(X; G), \ \mathcal{F}(L - T) < \delta\} \ ,$$

we get lower semicontinuity of $\hat{\mathbb{M}}$ on $\mathcal{F}_m(X; G)$.

Compactness and RectifiabilityTheorems

Compactness Theorem. [DHP] *Suppose X is a compact metric space and G is a complete normed group with closed balls being compact. For $R > 0$,*

$$K_R \ = \ \{T \in \mathcal{F}_m(X;G) \ : \ \hat{\mathbb{M}}(T) + \hat{\mathbb{M}}(\partial T) \leq R\} \text{ is } \mathcal{F} \text{ compact.}$$

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Rectifiability Theorem. [DH] *Any flat chain T of finite mass is rectifiable in case the group G contains no nonconstant Lipschitz curve*.*

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The *rectifiability* conclusion here gives the desired geometric character to the Plateau problem solutions. While this rectifiabiity is *not true* for $G = \mathbb{R}$ with the usual absolute value norm $|\ |$, it is true for each *group* norm $|\ |^\alpha$ for $0 \leq \alpha < 1$.

Plateau Problems

Corollary. *With X and G as in (A) and $T_0 \in \mathcal{F}_m(X;G)$ with $\hat{\mathbb{M}}(T_0) < \infty$,*

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Proof. Since $\partial(T - T_0) = 0$, the compactness theorem (A) implies that a minimizing sequence in $\mathcal{A}$ contains a subsequence $T_i = (T_i - T_0) + T_0$ that is $\mathcal{F}$ convergent to T_∞.

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The second conclusion is similar.

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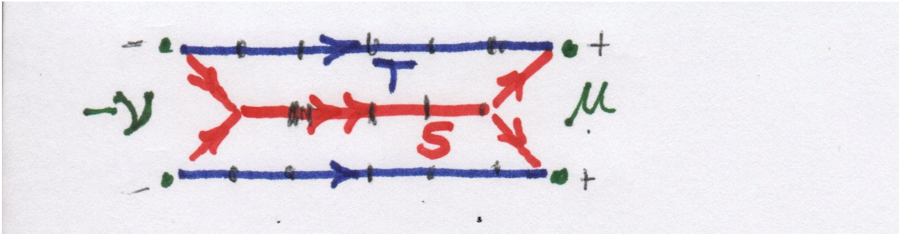
For $0 < \alpha < 1$, we define the norm $\|r\|_\alpha = |r|^\alpha$ for $r \in \mathbb{R}$. Then $(\mathbb{R}, \|\cdot\|_\alpha)$ *does* satisfy condition * . Also "merging" paths in T may reduce the corresponding mass $\mathbb{M}_\alpha(T)$.

Shared Transport Paths (mail, biology, economics, etc.)

We may connect two probability measures μ, ν in $\mathbb{R}^n$ by choosing $T \in \mathcal{F}_1(\mathbb{R}^n, G)$ with $\partial T = \mu - \nu$.

For $0 < \alpha < 1$, we define the norm $\|r\|_\alpha = |r|^\alpha$ for $r \in \mathbb{R}$. Then $(\mathbb{R}, \|\cdot\|_\alpha)$ *does* satisfy condition * . Also "merging" paths in T may reduce the corresponding mass $\mathbb{M}_\alpha(T)$.

Example.



$$\mathbb{M}_{\frac{1}{2}}(T) \;=\; 1 \cdot (6+6) \;>\; 1 \cdot 4\sqrt{2} + \sqrt{2} \cdot 4 \;=\; \mathbb{M}_{\frac{1}{2}}(S).$$

$\mathbb{M}_\alpha$ Minimizers

Corollary.(Q. Xia,2003) *There exists a* $\mathbb{M}_\alpha$ *minimizing* $T \in \mathcal{R}_1(\mathbb{R}^n, \mathbb{R})$ *with* $\partial T = \mu - \nu$.

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Key here is that, in contrast to the $\alpha = 1$ case of Almgren, one has

Graphical Approximation Lemma *Near a point having a single multiplicity Q tangent plane, the minimizer is close in measure and mass to a Q multiple of a single-valued Lipschitz function*