

Rencontres d'Analyse

Winter school : Nonlinear function spaces in Mathematics
and Physical Sciences

Lyon, December 14-18, 2015

Sobolev spaces into manifolds: a toolbox

Augusto C. Ponce

Sobolev maps with values into smooth manifolds can be defined in two non-equivalent ways: as Sobolev functions whose target is a manifold or by completion of smooth maps under the Sobolev norm. We explain why these approaches may lead to different nonlinear function spaces, depending on the topology of the target.

In some cases involving spheres, the obstruction can be detected using an elegant tool: the distributional Jacobian that quantifies the strength of topological singularities of a Sobolev map.

We explain some classical tools that can be used to investigate these questions

Lecture 1 How to define Sobolev spaces with values into a manifold?

Contributions: p. 10A

Two approaches of Sobolev spaces (real or vector-valued)

$$W^{1,p}(\mathbb{B}^m; \mathbb{R}^n) = \{ u \in L^p : Du \in L^p \} \quad \text{Du in the sense of distributions}$$

$H^{1,p}(\mathbb{B}^m; \mathbb{R}^n) = \text{completion of } C^\infty(\overline{\mathbb{B}^m}) \text{ with respect to } d_{W^{1,p}}$ distance

$$d_{W^{1,p}}(u, v) = \|u - v\|_{L^p} + \|Du - Dv\|_{L^p}.$$

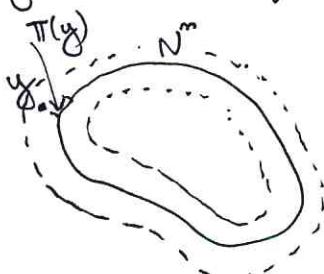
A classical property ensures that $H = W$.

Given a compact manifold $N^m \subset \mathbb{R}^N$, we may define W' and H accordingly:

$$W'^p(\mathbb{B}^m; N^m) = \{ u \in W^{1,p}(\mathbb{B}^m; \mathbb{R}^N) : u(x) \in N^m \text{ a.e.} \}$$

$H^{1,p}(\mathbb{B}^m; N^m) = \text{completion of } C^\infty(\mathbb{B}^m; N^m) \text{ with respect to } d_{W'^p}$

Compactness of N^m ensures a projection Π from a tubular neighborhood of N^m .



Approach to prove $H = W$ is based on convolution $f_R * u$.

If $p > m$, then we have uniform convergence and take $\Pi(f_R * u)$.

We do have $d_{W^{1,p}}(\Pi(p_k * u), u) \rightarrow 0$

Thm (Schoen-Weinbeck) For every $p \geq m$, $H^{\prime p} = W^{\prime p}$.

continuous

Proof Delicate case $p=m$: lack of Sobolev-Morrey imbedding

Use Poincaré inequality:

$$\int_{B(x; r)} |u - g_u|^p \leq C r^p \int_{B(x; r)} |Du|^p = C \int_{B(x; r)} |Du|^p.$$

Thus,

$$d_{\mathbb{R}^N}(N^m, g_u) \leq C \int_{B(x; r)} |Du|^m \rightarrow 0 \quad \text{uniformly as } r \rightarrow 0.$$

Similarly,

$$d_{\mathbb{R}^N}(N^m, p_k * u) \rightarrow 0.$$

We may thus use $\Pi(p_k * u)$ as before.

⚠ We do not pretend that $p_k * u$ converges uniformly to u ; $p_k * u$ is uniformly close to N^m .

Property still holds in VMO spaces (Boggi-Nirenberg)

⚠ Argument relies on the compactness of N^m .

Counter-example (only mention)



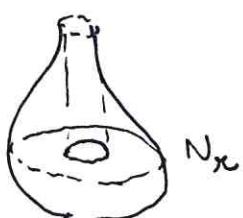
Take a bottle with infinite thin neck.

$u: B^m \rightarrow N^m$ projection over the bottle,
 $u \in W^{1,m}$ for a suitable bottle.

If $(\varphi_k)_{k \in \mathbb{N}} \rightharpoonup u$ in $d_{W^{1,m}}$, then for a.e. $x > 0$

$$\varphi_k \underset{\partial B_x}{\longrightarrow} u \quad \text{uniformly.}$$

May assume that $\varphi_k = u$ on ∂B_x to simplify



By the Brzmer degree, $\varphi_k(B_x)$ covers
 the lower part N_x of the bottle.

By the area formula, we thus have

$$\begin{aligned} H^m(N_x) &= H^m(\varphi_k(B_x)) \leq \int\limits_{B_x} |\det(D\varphi_k)| \\ &\leq \int\limits_{B_x} |D\varphi_k|^m \end{aligned}$$

Thus, $H^m(N_x) \leq \int\limits_{B_x} |Du|^m$; we get a contradiction as $x \rightarrow 0$.

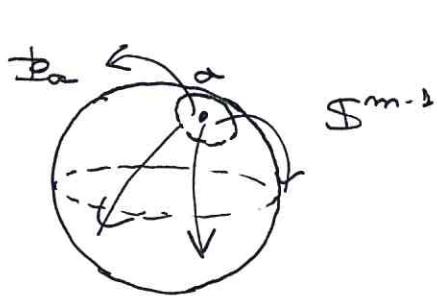
⚠ Analytical obstruction only for integer exponents.

What happens if $p < m$?

Consider the model case $W^{1,p}(\mathbb{B}^m; \mathbb{S}^{m-1})$:

~~Strategy~~ ^{from} Bethuel-Zhang: Fubini inspired by Federer-Fleming,
Hardt - ~~something~~, F.H. Lin

Trick: Given $\alpha \in \mathbb{S}^{m-1}$ and $\epsilon > 0$ small, consider



a Lipschitz map

$$\underline{\Phi}_\alpha: \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$$

$$\underline{\Phi}_\alpha(x) = x \text{ if } x \notin B(\alpha, \epsilon)$$

$$\underline{\Phi}_\alpha(B(\alpha, \epsilon)) \subset \mathbb{S}^{m-1} \setminus B(\alpha, \epsilon)$$

$$|D\underline{\Phi}_\alpha(x)| \leq \frac{C}{\epsilon}.$$

The set $\underline{\Phi}_\alpha(\mathbb{S}^{m-1})$ is diffeomorphic to a ball in \mathbb{R}^{m-1} .

Given $u \in W^{1,p}(\mathbb{B}^m; \mathbb{S}^{m-1})$, the function $\underline{\Phi}_\alpha \circ u$ has an image contained in a set with trivial topology.

How do we choose α correctly?

Note that $\forall \alpha$, $|D(\underline{\Phi}_\alpha \circ u)| \leq \frac{C}{\epsilon} |Du|$.

We have

$$D(\bar{\chi}_\alpha \circ u) - Du = 0 \quad \text{if } u(x) \notin B(\alpha; \epsilon)$$

$$\begin{aligned} |D(\bar{\chi}_\alpha \circ u) - Du| &\leq \left(\frac{C}{\epsilon} + 1 \right) |Du| \quad \text{if } u(x) \in B(\alpha; \epsilon) \\ &\leq \frac{C}{\epsilon} |Du| \end{aligned}$$

Thus,

$$\int_{B^m} |D(\bar{\chi}_\alpha \circ u) - Du|^p \leq \frac{C}{\epsilon^p} \int_{\{d(u(x), \alpha) < \epsilon\}} |Du|^p dx$$

By Fubini's theorem,

$$\begin{aligned} \int_{S^{m-1}} \left(\int_{B^m} |D(\bar{\chi}_\alpha \circ u) - Du|^p dx \right) d\alpha &\leq \\ &\leq \frac{C}{S_m \epsilon^p} \int_{B^m} |Du(x)|^p \underbrace{\left(\int_{\{d(u(x), \alpha) < \epsilon\}} d\alpha \right) dx}_{\text{volume of ball centered at } u(x), \text{ radius } \epsilon} \end{aligned}$$

$$\leq C \epsilon^{m-p-1} \int_{B^m} |Du|^p .$$

Choose $\alpha_\varepsilon \in \mathbb{S}^{m-1}$ such that

$$\int_{B^m} |D(\Phi_{\alpha_\varepsilon} u) - Du|^p dx \leq C \varepsilon^{m-p-1} \int_{B^m} |Du|^p$$

$\rightarrow 0$ if $p < m-1$
bdd if $p = m-1$

We can now approximate $\Phi_{\alpha_\varepsilon} u$ by smooth maps

Summarizing : $H^{1,p} = W^{1,p}$ if $p < m-1$
Theorem (Bethuel-Zhang) • weak sequential density if $p = m-1$
not stated in BZ
but fair to attribute

Left $m-1 \leq p < m$

Counter-example Hedgehog function

$$u: B^m \rightarrow \mathbb{S}^{m-1}$$

$$u(x) = \frac{x}{|x|}$$

$$|Du| \leq \frac{1}{|x|} \in L^p(B^m) \text{ for } p < m.$$

thus, $u \in W^{1,p}(B^m; \mathbb{S}^{m-1})$ for every $p < m$.

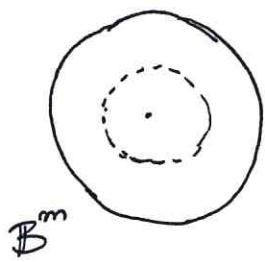
Theorem (Schoen-Uhlenbeck)

$u \notin H^{1,p}$ for every $p \geq m-1$

Proof By contradiction, assume that

$$\varphi_n \rightarrow u \text{ in } W^{1,p}$$

where $\varphi_n \in C^\infty(B^m; \mathbb{S}^{m-1})$.



By polar coordinates,

$$\int_0^1 \left(\int_{\partial B_x} |D\varphi_n - Du|^p \right) dx \rightarrow 0.$$

L^1 convergence with respect to x : There exists a subsequence (φ_{n_j}) such that

$$\int_{\partial B_x} |D\varphi_{n_j} - Du|^p \rightarrow 0 \quad \text{for a.e. } x.$$

If $p > m-1$, $\varphi_{n_j} \rightarrow u$ uniformly in ∂B_x

$$\deg(\varphi_{n_j}|_{\partial B_x}) = 0$$

$$\deg(u|_{\partial B_x}) = 1 \quad \text{Contradiction.}$$

Case $p = m-1$: continuity of \deg in VMO spaces.

and Fatou's lemma

Remark : Weak sequential density \rightarrow previous argument gives
 for a.e. $x > 0$ a subsequence $(\varphi_{n_j})_{j \in \mathbb{N}}$ bdd in $W^{1,p}(\partial B_x)$
 \Rightarrow equicontinuity by Sobolev-Morrey.

Summarizing:

$$W^1 P(\mathbb{B}^m; \mathbb{S}^{m-1})$$

p	Strong density	weak seq. density
$p \geq m$	Yes	\Rightarrow Yes
$m-1 < p < m$	No	\Leftarrow No
$p = m-1$	No	Yes
$p < m-1$	Yes	\Rightarrow Yes

What kind of topological obstruction should we expect for $m-1 \leq p < m$?

Only point singularities:

Thm (Bethuel-Zhang) $C^\infty(\overline{\mathbb{B}^m} \setminus \{\alpha, \dots, \alpha_l\}; \mathbb{S}^{m-1})$, $l \in \mathbb{N}$
is dense in $W^1 P(\mathbb{B}^m; \mathbb{S}^{m-1})$. for $p < m$.

Proof For $|\alpha| \leq 1/10$, consider a smooth map

$\underline{\Phi}_\alpha : \mathbb{B}^m \setminus \{\alpha\} \rightarrow \mathbb{S}^{m-1}$ such that

$$\underline{\Phi}_\alpha(z) = \frac{z}{|z|} \quad \text{if } |z| \geq 1/2$$

$$|\underline{\Phi}_\alpha(z)| \leq \frac{C}{|z-\alpha|} \quad \text{if } |z| < 1/2$$

(take $\underline{\Phi}_\alpha(z) = \frac{z-\alpha}{|z-\alpha|}$
in a neighborhood of α)

Take a sequence $(u_k)_{k \in \mathbb{N}}$ in $C^\infty(\mathbb{B}^m; \mathbb{R}^n)$ converging to u in $W^1 P$
Then

$$|\underline{\Phi}(u_k) - \underline{\Phi}u| \begin{cases} = |\underline{\Phi}\frac{u_k}{|u_k|} - \underline{\Phi}u| & \text{if } |u_k| \geq 1/2 \\ \leq C \frac{|\underline{\Phi}u_k|}{|u_k - \alpha|} + |\underline{\Phi}u| & \text{if } |u_k| < 1/2. \end{cases}$$

Since $u_k \rightarrow u$ in L^p , we have $\|\{u_k\}_{k=1}^\infty\|_p \rightarrow 0$.

We choose α using Fubini's theorem:

$$\int_{B(0,1)} \left(\int_{\{|u_k| < 1/2\}} \frac{|Du_k|^p}{|u_k - \alpha|^p} dx \right) d\alpha \leq C \int_{\{|u_k| < 1/2\}} |Du_k|^p$$

for $p < m$.

a regular value

Choose α_k such that

$$\int_{\{|u_k| < 1/2\}} \frac{|Du_k|^p}{|u_k - \alpha_k|^p} dx \leq C \int_{\{|u_k| < 1/2\}} |Du_k|^p \rightarrow 0.$$

The function $\Phi_{\alpha_k} \circ u_k$ is smooth except at finitely many points.

This approach will be pursued in lecture 3.

Contributions :

Density : • strong density :

→ Schoen-Wilkenbeck, Bethuel-Zhang, Bethuel, Hajeray,
Hang-Lim
(global obstruction)

• higher order :

→ BPVS, Gartel-Nerf, Hardt-Rivière

• fractional

→ Escobedo, Bethuel (trace), Brezis-Mironescu

• complete manifolds

→ Hajeray-Schinnerer, BPVS

• metric spaces (density of Lipschitz functions)

→ Koenraad-Schoen, Heinonen-Koskela-Shanmugalingam-Tyson

Hajeray

• weak density

→ Bethuel-Zhang, Bethuel, Hajeray, Hang-Lim,
Pazyd, Pazyd-Rivière, Hardt-Rivière,
Bethuel (counter-example)

For a compact manifold N^m :

Thm (Bethuel) For every $1 \leq p < m$, we have

$$H^{1,p} = W^{1,p} \text{ iff } \pi_{\lfloor l_p \rfloor}(N^m) \cong \{\text{id}\}.$$

$\lfloor l_p \rfloor$: integer part of p . (\Rightarrow) based on Schoen-Uhlenbeck

The topological property $\pi_e(N^m) \cong \{\text{id}\}$ does not require any deep knowledge of algebraic topology:

for every continuous map $f: S^l \rightarrow N^m$, there exists a continuous extension $F: B^{l+1} \rightarrow N^m$,

Example: $l=0$: path connectedness

$$\pi_e(S^m) \cong \{\text{id}\} \text{ for every } l < m$$

$$\pi_{\lfloor l \rfloor}(S^m) \neq \{\text{id}\}$$

Counterparts for $W^{s,p}$ $s \in \mathbb{N}$ obtained by BPVS
 $0 < s < 1$ Brezis-Mironescu

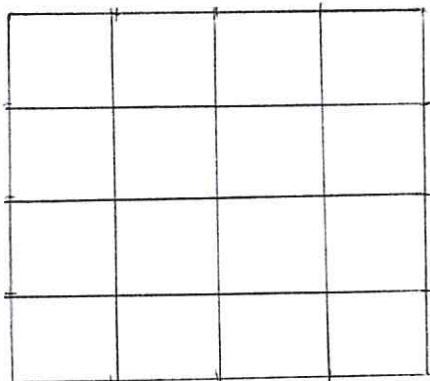
In these cases, equivalence $H=W$ involves $\pi_{\lfloor l_p \rfloor}(N^m) \cong \{\text{id}\}$.

Lecture 2 Toolbox for $H=W$

We present a proof of Besicovitch's theorem that can be adapted to integer orders.

Assume $m-1 < p < m$.

and $\text{Tr}_{m-1}(N^m) \leq \{0\}$.



Replace B^m by Q^m .

Divide Q^m into cubes Q_x of radius x .

Take $\rho \in C_c^\infty(Q_1)$ and consider the convolution

$$\rho_x * u \text{ where } \rho_x(x) = \frac{1}{x^m} \rho\left(\frac{x}{x}\right).$$

Tool 1 Convolution

Identify cubes where $\rho_x * u$ successfully approximate u .

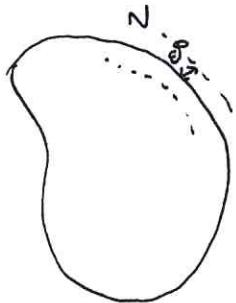
Use Poincaré inequality.

$$\int_{Q_x^m} |u - \rho_x * u|^p \leq C x^p \int_{Q_{2x}^m} |Du|^p$$

Proof : - fundamental thm
- or contradiction for $x=1$.

Hence

$$d(N^m, \rho_{x^*} u)^P \leq \sum_{x^{m-p}} \int_{Q_{2x}^m} |Du| P.$$



Take a tubular neighborhood of N^m , equipped with a smooth projection.

If $d(N^m, \rho_{x^*} u) \leq \varepsilon$, we may project $\pi(\rho_{x^*} u)$.

Define a good cube Q_x if

$$\sum_{x^{m-p}} \int_{Q_{2x}^m} |Du| P \leq \varepsilon^p$$

Bad cube otherwise.

G_x : class of good cubes; B_x : bad cubes.

Terminology from Beskane.

⚠ There are not many bad cubes.

Total volume $\#B_x \times x^m \rightarrow 0$ as $x \rightarrow 0$.

We first count the number of bad cubes $\# B_x$.

$$\text{A bad cube : } \frac{C}{x^{m-p}} \int_{Q_{2x}} |Du|^p > \varepsilon^p.$$

The cubes Q_x are disjoint; the cubes Q_{2x} have finitely many overlaps Θ_N :

$$\sum_{Q_x \in B_x} \chi_{Q_{2x}} \leq \Theta_N$$

$$\sum_{Q_x \in B_x} \chi_{Q_{2x}} |Du|^p \leq \Theta_N |Du|^p$$

Thus,

$$\frac{\varepsilon^p x^{m-p}}{C} \# B_x \leq \sum_{Q_x \in B_x} \int_{Q_{2x}} |Du|^p \leq \Theta_N \int_{\mathbb{R}^m} |Du|^p$$

We deduce that

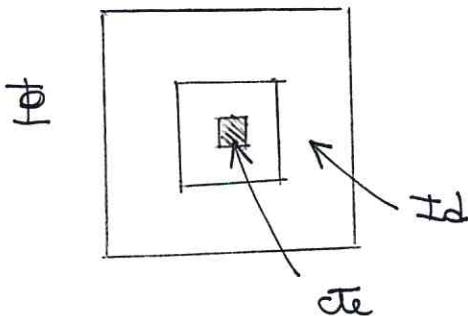
$$\# B_x \leq \frac{C}{x^{m-p}}.$$

Total volume of bad cubes:

$$\# B_x \times x^m \leq C x^p \rightarrow 0 \text{ as } x \rightarrow 0.$$

Tool 2 Opening

Based on Brezis-Yu Li



Goal: Lemma For every v there exists $\underline{\Phi}$

$$v \in W^{1,p} \mapsto v \circ \underline{\Phi} \in W^{1,p}$$

where

$v \circ \underline{\Phi}$ is constant in a neighborhood of 0

$$\int |D(v \circ \underline{\Phi})|^p \leq C \int |Dv|^p.$$

Δ $\underline{\Phi}$ depends on v .

Nonlinear construction based on Fubini's thm.

Take $\underline{\Phi}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ smooth

$$\underline{\Phi}(z) = \begin{cases} z & \text{in } \mathbb{R}^m \setminus Q_{1/2} \\ 0 & \text{in } Q_{1/4} \end{cases}$$

Let

$$\underline{\Phi}_\alpha(z) = \underline{\Phi}(z+\alpha) - \alpha.$$

For $\alpha \in Q_{1/8}$, $\underline{\Phi}_\alpha = -\alpha$ in $Q_{1/8}$.

We have

$$|D(v \circ \underline{\Phi}_\alpha)|^{(x)} \leq C |Dv|(\underline{\Phi}(z+\alpha) - \alpha).$$

— 15 —

By Fubini's theorem,

$$\int_{Q_{1/8}} \left(\int_{Q_1} |Dv|^P (\Phi(x+\alpha) - \alpha) dx \right) d\alpha =$$

Fubini

$$= \int_{Q_1} \left(\int_{Q_{1/8}} |Dv|^P (\Phi(x+\alpha) - \alpha) d\alpha \right) dx$$

$$\begin{aligned} z &= x + \alpha \\ \alpha &= x - z \end{aligned}$$

$$= \int_{Q_1} \left(\int_{x+Q_{1/8}} |Dv|^P (\Phi(z) - x + z) dz \right) dx$$

Fubini

$$\begin{aligned} z \in x+Q_{1/8} &= \int_{Q_1} \left(\int_{(z-Q_{1/8}) \cap Q_1} |Dv|^P (\Phi(z) - x + z) dx \right) dz \\ x \in z-Q_{1/8} & \quad Q_1+Q_{1/8} \end{aligned}$$

$$y = \Phi(z) - x + z$$

$$= \int_{Q_1+Q_{1/8}} \left(\int_{(\Phi(z)-Q_{1/8}) \cap (\Phi(z)+z-Q_1)} |Dv|^P(y) dy \right) dz$$

$$\leq |Q_1+Q_{1/8}| \int_{Q_2} |Dv|^P \quad \text{for suitable choice of } \Phi.$$

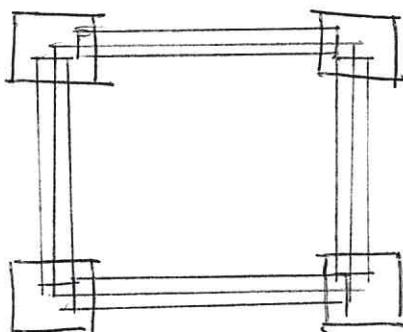
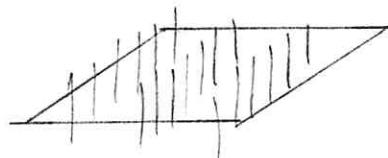
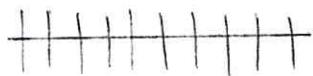
Thus,

$$\int_{Q_1 \setminus Q_2} |D(\nu \circ \tilde{\tau}_\alpha)|^p d\alpha \leq C \int_{Q_2} |D\nu|^p.$$

Choose α such that

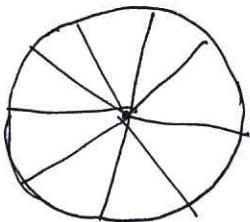
$$\int_{Q_2} |D(\nu \circ \tilde{\tau}_\alpha)|^p \leq C \int_{Q_2} |D\nu|^p.$$

We can similarly open along lines, planes, ...



If $m-1 < p < m$, function becomes continuous nearby $\partial\Omega$;
VMO if $p = m-1$.

Tool 3 Thickening (homogenization)



Lemma Let $p < m$.

Given $u \in W^{1,p}(\partial B_1; N^m)$, then for

$$v(x) = u\left(\frac{x}{|x|}\right)$$

satisfies

$$v \in W^{1,p}(B_1; N^m)$$

$$\text{and } \|Du\|_{W^{1,p}(B_1)} \leq C \|Dv\|_{L^p(\partial B_1)}.$$

Proof

$$|Du|(x) \leq |Du\left(\frac{x}{|x|}\right)| \frac{1}{|x|}$$

Thus

$$\begin{aligned} \int_{B_1} |Du|^p &\leq \int_{B_1} \left| Du\left(\frac{x}{|x|}\right) \right|^p \frac{1}{|x|^p} dx \\ &= \int_0^1 \left(\int_{\partial B_1} |Du|^p \right) \frac{x^{m-1}}{x^p} dx = \\ &= C_p \int_{\partial B_1} |Du|^p \quad \text{for } p < m. \end{aligned}$$

⚠ This construction creates a point singularity.

We use opening $\overset{\circ}{u}$ on $\partial\Omega_x$ of a bad cube and then propagate the values of $\rho_x * u$ using thickening.

No topological assumption has been used.

Tool 4 Removable singularity

Summarize tools 1-4
at the end
→ p. 18A

From Bethuel-Zhang (lemma 5).

Lemma Let $p < m$ and $u \in W^{1,p}(B_1^m; N^m)$, $0 < \delta < 1$.

If u is continuous on $B_1^m \setminus \{0\}$ and $u|_{\partial B_\delta^m} : \partial B_\delta^m \rightarrow N^m$ has a continuous extension $F : \overline{B_\delta^m} \rightarrow N^m$, then there exists $\tilde{u} \in W^{1,p}(B_\delta^m; N^m) \cap C^0(\overline{B_\delta^m})$ such that

$$\tilde{u} = u \text{ on } \overline{B_\delta^m} \setminus B_\delta^m$$

$$\int_{B_\delta^m} |Du|^p \leq C \int_{B_1^m} |Du|^p.$$

From an extension we can get another one with control on ^{the} energy.
This is the step where the topological assumption is used.

Proof We assume F smooth on B_1
 u smooth on $B_1 \setminus \{0\}$ and homogeneous

$$u(x) = u\left(\frac{x}{|x|}\right)$$

Take

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \notin B_\delta \\ F\left(\frac{x}{\delta}\right) & \text{if } x \in B_\delta \end{cases}$$

Then,

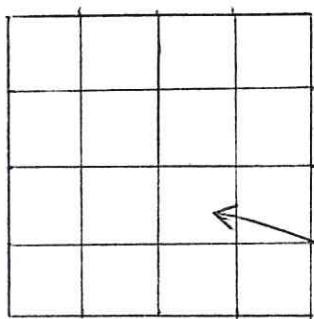
$$\int_{B_\delta^m} |D\tilde{u}|^p = \int_{B_\delta^m} |DF|^p \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

$W^{1,p}$

Argument implicitly relies on the fact that $\{0\}$ has zero capacity.

Implementation following BPVS (JEMs 2015)

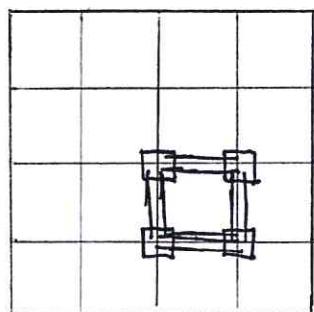
1



Divide \mathbb{Q}^m as a union of cubes \mathbb{Q}_x ;
identify good and bad cubes

bad cube

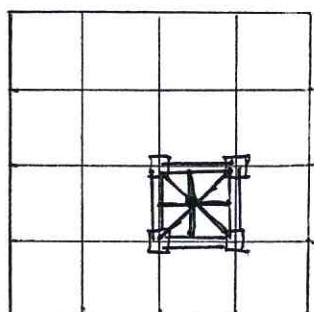
2



Successively apply opening on vertices ,
edges, faces of bad cubes

3 Convolution on good cubes ;
convolution parameter vanishes as we approach a bad cube
project back to manifold N^m .

4



Thickening (homogenization) on bad cubes

5 Removal of singularity (assuming $T_{m-1}(N^m) = \{0\}$)

Initial map u
Energy : Final map v continuous

$$\int_{\mathbb{Q}^m} |Du - Dv|^p \leq \underbrace{\int_{\mathbb{Q}^m} |D(p_0 + u) - Du|^p}_{\text{estimate from good cubes}} + C \underbrace{\int_{\cup \mathbb{Q}_x \atop \mathbb{Q}_x \text{bad}} |Du|^p}_{\text{estimate from bad cubes}}$$

Lecture 3 How to detect and quantify topological singularities?

Contributions: pp. 28-29

Detour to lifting problem:

Given $u: B^m \rightarrow S^1$, find $f: B^m \rightarrow \mathbb{R}$ such that $u = e^{if}$.

$$\begin{array}{ccc} & \mathbb{R} & \\ f \nearrow & \downarrow e^{it} & \\ B^m & \xrightarrow{u} & S^1 \end{array}$$

We identify $\mathbb{R}^2 \cong \mathbb{C}$; S^1 is parametrized by $t \mapsto e^{it}$.

Answer depends on functional spaces:

- if $u \in C^0$, then there exists $f \in C^0$
- if $u \in C^\infty$, then $f \in C^\infty$.

Note that

$$Du = e^{if} \cdot Df = u \cdot Df$$

Hence

$$|Du| = |Df|.$$

It is meaningful to ask the question for $W^{1,p}$ spaces.

Easy answer:

Prop There exists $f \in W^{1,p}(B^m; \mathbb{R})$ such that $u = e^{if}$ a.e.
iff $u \in H^{1,p}(B^m; S^1)$.

In view of the characterization $H = W$, lifting holds iff $p \geq 2$
(Bottau - Zheng).

We want another characterization of maps $H^1 P(B^m; \mathbb{S}^1)$.

Approach based on the Poincaré lemma (underlying differential equation).

If $u \in W^{1,p}(B^m; \mathbb{S}^1)$ and $u = e^{if}$ with $f \in W^{1,p}(B^m; \mathbb{R})$,

$$\boxed{df = \frac{1}{i} \bar{u} Du}$$

In terms of partial derivatives:

$$\partial_k f = \frac{1}{i} \bar{u} \partial_k u = u_1 \partial_k u_2 - u_2 \partial_k u_1 = u \times \partial_k u$$

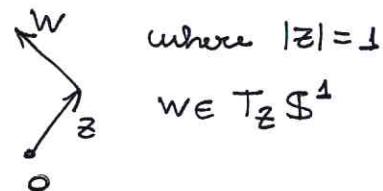
\uparrow
 only real part \uparrow
 vector product
in \mathbb{R}^2

In terms of differential forms:

$$\boxed{df = u^\# \omega_{\mathbb{S}^1}}$$

Background $\omega_{\mathbb{S}^1}$ volume form on \mathbb{S}^1

$$\omega_{\mathbb{S}^1}(z)[w] = z \times w$$



$u^\# \omega_{\mathbb{S}^1}$: pull back of $\omega_{\mathbb{S}^1}$
 \hookrightarrow differential form on B^m

$$u^\# \omega_{\mathbb{S}^1}(x)[v] = \underbrace{\omega_{\mathbb{S}^1}(u(x))}_{\in \mathbb{S}^1} \underbrace{[Du(x)[v]]}_{\in T_{u(x)} \mathbb{S}^1}$$

$x \in B^m$ $v \in \mathbb{R}^m$

Hence

$$\begin{aligned} u^\# \omega_{\mathbb{S}^1} &= \omega_{\mathbb{S}^1}(u) \left[\frac{\partial u}{\partial x_1} \right] dx_1 + \omega_{\mathbb{S}^1}(u) \left[\frac{\partial u}{\partial x_2} \right] dx_2 \\ &= u \times \frac{\partial u}{\partial x_1} dx_1 + u \times \frac{\partial u}{\partial x_2} dx_2 \end{aligned}$$

Poincaré lemma (L^p version)

Let $g = g_1 dx_1 + \dots + g_m dx_m$ with $g_j \in L^p(B^m)$.

There exists $f \in W^{1,p}(B^m)$ such that

$df = g$ iff $\underbrace{dg = 0}$ in the sense of distributions.

$$\partial_k g_j = \partial_j g_k \quad \forall j, k.$$

Proof by approximation; note that $dg = 0$ implies that
 $d(\rho * g) = 0$.

Conclusion

Let $\omega \in W^{1,p}(B^m; \mathbb{S}^1)$.

There exists $f \in W^{1,p}(B^m; \mathbb{R})$ such that $df = \omega^\# \omega_{\mathbb{S}^1}$ iff
 $d(\omega^\# \omega_{\mathbb{S}^1}) = 0$ in the
sense of
distributions.

Back to the lifting problem:

Theorem (Démongeol)

Let $\omega \in W^{1,p}(B^m; \mathbb{S}^1)$.

There exists $f \in W^{1,p}(B^m; \mathbb{R})$ such that $\omega = e^{if}$ iff $d(\omega^\# \omega_{\mathbb{S}^1}) = 0$.

Proof Solve $df = \omega^\# \omega_{\mathbb{S}^1}$.

$$\text{Then, } \partial_k f = \frac{1}{i} \bar{\omega} \partial_k \omega.$$

$$\text{Hence, } \partial_k (e^{-if} \omega) = 0 \text{ and this gives } e^{-if} \omega = c \bar{e} = e^{ic}.$$

Corollary

Let $u \in W^{1,p}(B^m; S^1)$

We have $u \in H^{1,p}$ iff $d(u^\# w_{S^1}) = 0$.

Remark: Assumption $d(u^\# w_{S^1}) = 0$ is always satisfied if $p \geq 2$.

This relies on an easy density argument in $W^{1,p}(B^m; \mathbb{R}^2)$.

Take a sequence $(u_k)_{k \in \mathbb{N}}$ in $C^\infty(B^m; \mathbb{R}^2)$, $u_k \rightarrow u$ in $W^{1,2}$.

For every $j, l \in \{1, \dots, m\}$,

$$\partial_j(u_k \times \partial_l u_k) - \partial_l(u_k \times \partial_j u_k) = 2 \partial_j u_k \times \partial_l u_k$$

Thus, as $k \rightarrow \infty$,

$$\partial_j(u \times \partial_l u) - \partial_l(u \times \partial_j u) = 2 \partial_j u \times \partial_l u.$$

Since $\partial_j u \parallel \partial_l u$, the RHS = 0.

Distributional jacobian

$m=2$ and more generally $W^{1,p}(B^m; S^{m-1})$ for $p \geq m-1$.

For $m-1 \leq p < m$ we expect topological point singularities.

Define

$$\text{Jac}(u) = d(u^\# w_{S^{m-1}}) \text{ in the sense of distributions}$$

$$\langle \text{Jac}(u), \varphi \rangle = - \int_{B^m} d\varphi \wedge u^\# w_{S^{m-1}} \quad \text{for every } \varphi \in C_c^\infty(B^m)$$

This is well-defined for $p \geq m-1$. Indeed,

$$u^* \omega_{S^{m-1}} [v_1, \dots, v_{m-1}] = \omega_{S^{m-1}}(u) [Du[v_1], \dots, Du[v_{m-1}]].$$

$$\in L^1(B^m)$$

Assume that $u \in C^\infty(B^m \setminus \{a\}; S^{m-1})$.

Note that

$$d(\varphi \wedge u^* \omega_{S^{m-1}}) = d\varphi \wedge u^* \omega_{S^{m-1}} + \varphi \wedge d(u^* \omega_{S^{m-1}})$$

and

$$d(u^* \omega_{S^{m-1}}) = u^* d\omega_{S^{m-1}} = 0$$

Thus, for every $\epsilon > 0$ small,

$$\begin{aligned} \int_{B^m \setminus B(a; \epsilon)} d\varphi \wedge u^* \omega_{S^{m-1}} &= \int_{B^m \setminus B(a; \epsilon)} d(\varphi \wedge u^* \omega_{S^{m-1}}) \\ &= - \int_{\partial B(a; \epsilon)} \varphi \wedge u^* \omega_{S^{m-1}} \end{aligned}$$

As $\epsilon \rightarrow 0$,

$$-\langle \text{Jac}(u), \varphi \rangle = -\varphi(a) \deg(u, a) \cdot \sigma_m$$

[Recall that $\deg(u|_{\partial B(a, \epsilon)}) = \frac{1}{\sigma_m} \int_{\partial B(a, \epsilon)} u^* \omega_{S^{m-1}}$]

where σ_m is the area of S^{m-1}
 $(H^{m-1}(S^{m-1}))$

Hence, $\text{Jac}(u) = \sigma_m \deg(u, a) \delta_a$.

More generally, if $u \in C^\infty(\overline{B^m} \setminus \{\alpha_1, \dots, \alpha_\ell\}; \mathbb{S}^{m-1})$
with α_i distinct,

$$\text{Jac}(u) = \sigma_m \sum_{i=1}^{\ell} \deg(u, \alpha_i) \delta_{\alpha_i}$$

Assuming that $u = e$ is constant on ∂B^m , we have

$$\sum_{i=1}^{\ell} \deg(u, \alpha_i) = 0,$$

hence

$$\text{Jac}(u) = \sigma_m \sum_{i=1}^n (\delta_{p_i} - \delta_{m_i}).$$

→ P-24 A, B for another representation

Theorem (Brugis - Mironeanu) (*)

For every $u \in W_e^{1,p}(B^m; \mathbb{S}^{m-1})$, $m-1 \leq p < m$, there exist

$(p_i)_{i \in \mathbb{N}}, (m_i)_{i \in \mathbb{N}}$ such that $\sum_{i=0}^{\infty} d(p_i, m_i) < +\infty$ and

$$\text{Jac}(u) = \sigma_m \sum_{i=0}^{\infty} (\delta_{p_i} - \delta_{m_i})$$

The distribution in the right-hand side is well defined
in the class of Lipschitz functions:

$$T = \sum_{i=0}^{\infty} (\delta_{p_i} - \delta_{m_i})$$

$$|\langle T, \varphi \rangle| \leq [\varphi]_{\text{Lip}} \sum_{i=0}^{\infty} d(p_i, m_i)$$

for every $\varphi \in C_c^\infty(B^m)$ or even $\varphi \in \text{Lip}_0(B^m)$

$[\varphi]_{\text{Lip}}$ is a norm

By definition of the integral of a differential form,

$$\begin{aligned} \int_{\mathbb{B}^m} d\varphi \wedge u^\# \omega_{\mathbb{S}^2} &= \\ &= \int_{\mathbb{B}^m} \langle d\varphi(x) \wedge u^\# \omega_{\mathbb{S}^2}(x), n(x) \wedge e_2(x) \wedge \dots \wedge e_m(x) \rangle dx \end{aligned}$$

By definition of differential

$$d\varphi(x)[v] = D\varphi(x)[v] = \begin{cases} |\nabla\varphi(x)| & \text{if } v = n(x) \\ 0 & \text{if } v = e_i, i \in \{2, \dots, m\} \text{ since } v \in \ker D\varphi(x). \end{cases}$$

Hence,

$$\int_{\mathbb{B}^m} d\varphi \wedge u^\# \omega_{\mathbb{S}^2} = \int_{\mathbb{B}^m} |\nabla\varphi(x)| \langle u^\# \omega_{\mathbb{S}^2}(x), e_2 \wedge \dots \wedge e_m \rangle dx$$

$$\xrightarrow{\text{co-area formula}} = \int_{\mathbb{R}} \left(\int_{\varphi^{-1}(ty)} u^\# \omega_{\mathbb{S}^2} \right) dy$$

[Computation above taken from Bourbaki - Manifolds]

Prop (Bougain - Brezis - Mironescu)

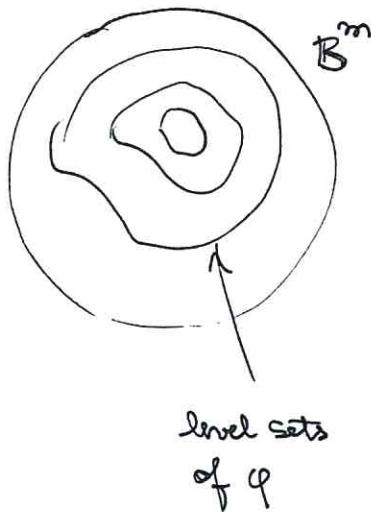
$u: \overline{B^m} \rightarrow S^{m-1}$ smooth except at finitely many points

Then,

$$\langle \text{Jac}(u), \varphi \rangle = \mathcal{C}_{m-1} \int_{\mathbb{R}} \deg(u, \varphi^{-1}(y_i)) dy$$

for every $\varphi \in C_c^\infty(\overline{B^m})$

Rk Using a density argument, formula holds for $u \in W^{1,m-1}(\overline{B^m}; S^{m-1})$.



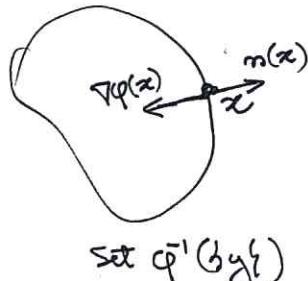
We need the co-area formula:

For every $f \in L^1(\overline{B^m})$,

$$\int_{B^m} f |\nabla \varphi| = \int_{\mathbb{R}} \left(\int_{\varphi^{-1}(y_i)} f d\sigma \right) dy$$

Surface measure
 \mathcal{H}^{m-1}

Take unit normal vector



$$n(x) = - \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|}$$

complete to get an orthonormal basis

n, e_2, \dots, e_m positively oriented in \mathbb{R}^m ;

where $e_i \in T_x(\varphi^{-1}(y_i)) = \ker(D\varphi(x))$

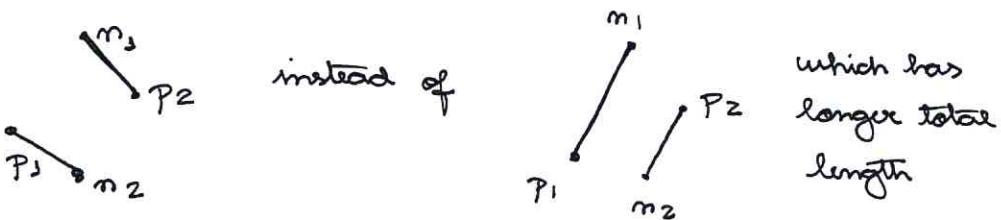
Come back to $T = \sum_{i=1}^k (\delta_{\tilde{p}_i} - \delta_{\tilde{m}_i})$

We actually have

$$\|T\|_{(\text{Lip}_0)} = \inf \left\{ \sum_{i=1}^k d(\tilde{p}_i, \tilde{m}_i) : T = \sum_{i=1}^k (\delta_{\tilde{p}_i} - \delta_{\tilde{m}_i}) \right\}$$

[obtained by permutation]

Optimal transportation problem by Brenier-Carrier-Lieb based on Kantorovich: minimal connection between positive and negative points.



In view of the formula

$$\langle \text{Jac}(u), \varphi \rangle = - \int_{B^m} d\varphi \wedge (u^\# \omega^{m-1})$$

we have continuity of the Jacobian as a map

$$\text{Jac} : W_e^{1,p}(B^m; S^{m-1}) \rightarrow (\text{Lip}_0(B^m))'$$

Indeed, we may write

$$\langle \text{Jac}(u), \varphi \rangle = - \int_{B^m} \nabla \varphi \cdot j(u)$$

where $j(u)$ is a vector field whose components are

$$\det(u, \partial_1 u, \dots, \hat{\partial_j u}, \dots, \partial_m u),$$

whence $u \mapsto j(u)$ is continuous \Rightarrow for $p \geq m-1$.

To prove theorem (*), take a sequence $(u_k)_{k \in \mathbb{N}}$ in

$C_c^\infty(B^m; \mathbb{H}^{n-1}; \mathbb{S}^{m-1})$ converging fast to u in $W^{1,p}$.

Write

$$\text{Jac}(u) = \text{Jac}(u_0) + \sum_{k=1}^{\infty} (\text{Jac}(u_k) - \text{Jac}(u_{k-1})).$$

By continuity of Jac at u , we may assume that

$$\sum_{k=1}^{\infty} \|\text{Jac}(u_k) - \text{Jac}(u_{k-1})\|_{(\text{Lip}_e)} < +\infty.$$

and write

$$\text{Jac}(u_k) - \text{Jac}(u_{k-1}) = \sum_{i=1}^{l_k} (\delta_{p_{i,k}} - \delta_{m_{i,k}})$$

in an optimal way. This gives the representation of $\text{Jac}(u)$.

Theorem Take $u \in W^{1,p}(B^m; \mathbb{S}^{m-1})$, $m-1 \leq p < m$.

We have $u \in H^{1,p}$ if and only if $\text{Jac}(u) = 0$.

Case $p = m-1$: Bethuel based on a removing dipole technique

$m-1 < p < m$: Bethuel - Coron - Demengel - Helein.

based on removable singularity

$\left \begin{array}{l} \text{this case is usually misquoted and attributed to} \\ \text{Bethuel alone;} \end{array} \right.$
$\left \begin{array}{l} \text{the case } p > m-1 \text{ is never mentioned in Bethuel's paper} \\ \text{in Ann IHP.} \end{array} \right.$

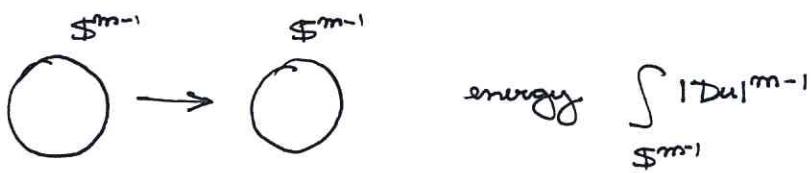
for $p = m^{-1}$

—27—

Proof based on this:

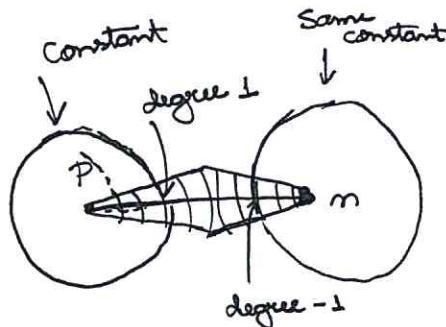
Dipole technique

$$u: \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$$
$$x \mapsto x$$



concentration without changing energy

(using stereographic projection)

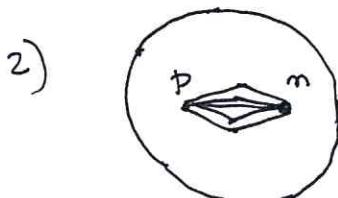


$$\text{energy} \sim d(p, m)$$

How to remove the singularity of a map $u \in C^\infty(B^m \setminus \{p, m\}; \mathbb{S}^{m-1})$
such that $\deg(u, p) = +1$ and $\deg(u, m) = -1$?



Take a small conical region and modify
 u so as to get a map which is constant there.
↳ small change of energy.



Add a dipole with inverted poles
We get a new map \tilde{u} such that

$$\deg(\tilde{u}, p) = +1 - 1 = 0$$

$$\deg(\tilde{u}, m) = -1 + 1 = 0$$

We may remove
topologically trivial
singularities at
 p and m .

Energy increased by a factor $d(p, m)$:

$$\int_{B^m} |D\tilde{u} - Du|^{m-1} \sim d(p, m).$$

Contributions

Lifting: Bethuel-Zhang ($t \mapsto e^{it}$)

↳ Bethuel-Chiron

(containing Mucci and Ball-Zarnescu $\pi: \mathbb{S}^m \rightarrow \mathbb{RP}^m$)

• Fractional Sobolev spaces

→ Bourgain-Brezis-Mironescu, Nguyen H.-M.

book: BM

• BV: Daala-Ignat, Merlet

Some contributions

Jacobian : • distributional :

↳ Ball, Bregis-Coron-Lieb, Giacinta-Modica-Souček
↑ ↑
elasticity manifolds

• variants :

↳ Zhou, Hardt-Rivière, Bethuel-Coron-Demengel-Hlein

• existence of prescribed singularities :

↳ Bethuel, Bourgoin, Alberti-Baldo-Oxlandi
* *

*: context of minimal connections

• characterization of $H^{1,p}$:

Bethuel: $W^{1,m-1}(B^m; \mathbb{S}^{m-1})$

BCDH: $W^{1,p}(B^m; \mathbb{S}^{m-1}) \quad m-1 \leq p < m$

| Pakzad, Taubes-Friese
| Giacinta-Mucci $W^{1,2}(B^m; \mathbb{S}^2)$
| Giacinta-Modica-Souček

Zhou: $W^{1,3}(B^m; \mathbb{S}^2)$

Hardt-Rivière: $W^{2,2}(B^5; \mathbb{S}^3)$

Demengel: $W^{1,p}(B^m; \mathbb{S}^1) \quad 1 \leq p < 2$.

• Jacobian in fractional $W^{s,p}$

↳ Hang-Lin, Bourgain-Bregis-Mironescu,
Bourgain-Mironescu