Branched transportation and singularities of Sobolev maps between manifolds
Part I : Introduction, motivation, topology

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Definition

Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{N}$ embedded in $\mathbb{R}^\ell$. For $1 \leq p < \infty$ we consider the Sobolev space with pointwise constraint

$$W^{1,p}(\mathcal{M},\mathcal{N}) = \{u \in W^{1,p}(\mathcal{M},\mathbb{R}^\ell),\ u(x) \in \mathcal{N} \text{ for a.e } x \in \mathcal{M}\}.$$

A typical example to keep in mind is $\mathcal{M} = \mathbb{B}^3$, $\mathcal{N} = \mathbb{S}^2$ so that

$$W^{1,p}(\mathbb{B}^3,\mathbb{S}^2) = \{u = (u_1,u_2,u_3) \in W^{1,p}(\mathbb{B}^3,\mathbb{R}^3), |u(x)| = 1 \text{ for a.e } x = \in \mathbb{B}^3\}.$$

Motivations

This spaces are the natural settings for **models** in calculus of variation:

- In a liquid crystal, molecules are free to move but oriented locally at point x of the container $\Omega \subset \mathbb{R}^3$ in a common direction given by a unit vector $u(x)$, yielding a map $u : \Omega \to \mathbb{S}^2$. In the Oseen-Frank model stable configuration minimize or are critical points of E, which in its simplest form reduces

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2. \tag{1}$$

  The natural set for E is $W^{1,2}(\Omega, \mathbb{S}^2)$, the constraint $|u| = 1$ inducing a nonlinear term in the Euler-Lagrange equation for E, the equation for *harmonic maps into the sphere*, which writes $-\Delta u = u|\nabla u|^2$.
- More generally, Harmonic maps are critical points of the Dirichlet energy in $W^{1,2}(\mathscr{M}, \mathscr{N})$ and are a natural extension of the notion of *harmonic functions*.

Remark

Since weak topology and weak convergence is of special importance for calculus of variation this notion will also be one of our main focus.

Topology and Sobolev spaces

In our discussion we will often face the following facts :

- Topology deals usely with continous maps
- Calculus of variations favors Sobolev spaces.

For Instance, it is not obvious to minimize energies in homotopy classes.

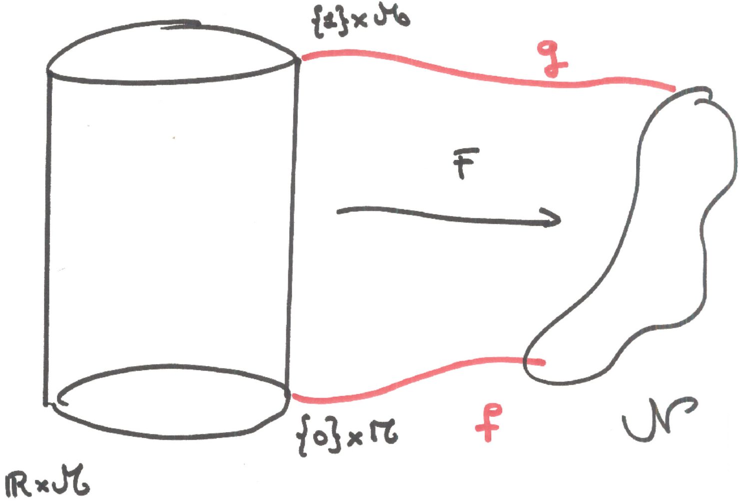
We next review some elementary facts in Topology, starting with homotopy classes.

Homotopy classes

We consider here only continuous maps, i. e. maps in $C^0(\mathscr{M},\mathscr{N})$.
Recall that two maps f and g are said to be in the same homotopy class in there exist a map $F \in C^\infty([0,1]\times\mathscr{M},\mathscr{N})$ called a deformation of f to g such that

$$\begin{cases} F(0,x) = f(x) \text{ for } x \in \mathscr{M} \\ F(1,x) = g(x) \text{ for } x \in \mathscr{M} \end{cases}$$

yielding an equivalence relation in $C^0(\mathscr{M},\mathscr{N})$. The corresponding equivalent classes are called **homotopy classes**.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Homotopy groups

In the case the domain $\mathscr{M}$ in a sphere $\mathbb{S}^m$ the homotopy classes of $C^0(\mathbb{S}^m, \mathscr{N})$ form a group called the m- homotopy group of $\mathscr{N}$ and denoted $\pi_m(\mathscr{N})$.

A group action can be defined using gluing.

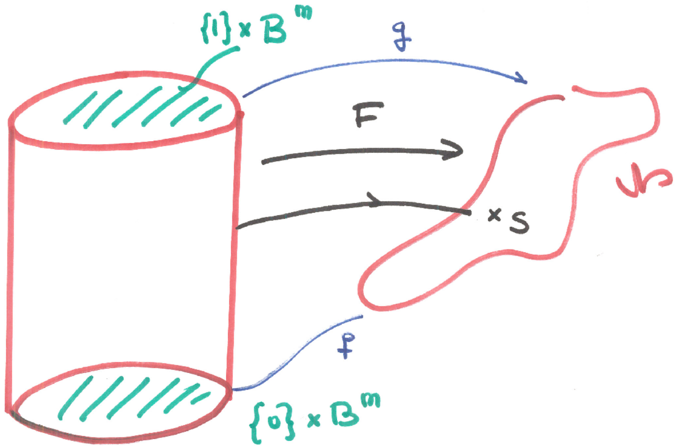
Homotopy classes with boundary

Instead of the sphere $\mathbb{S}^m$ we may consider the ball $\mathbb{B}^m$, but prescribing the valued of the map to be fixed boundary on the boundary. Let S be an arbitrary point on $\mathcal{N}$, assumed to be connected. We consider the set

$$C^0_{\mathrm{S}}(\mathbb{B}^m, \mathcal{N}) = \{u \in C^0(\mathbb{B}^m, \mathcal{N}), u(x) = \mathrm{S} \text{ on } \partial\mathbb{B}^m\}.$$

On $C^0_{\mathrm{S}}(\mathbb{B}^m, \mathcal{N})$ we may define as abovehomotopy classes: it suffices to impose additional conditions on the deformation F that

$$\begin{cases} F(0,x) = f(x) \text{ for } x \in \mathbb{B}^m \\ F(1,x) = g(x) \text{ for } x \in \mathbb{B}^m \\ F(x,t) = \mathrm{S} \text{ for any } t \in [0,1] \text{ and } x \in \partial B^m\}. \end{cases}$$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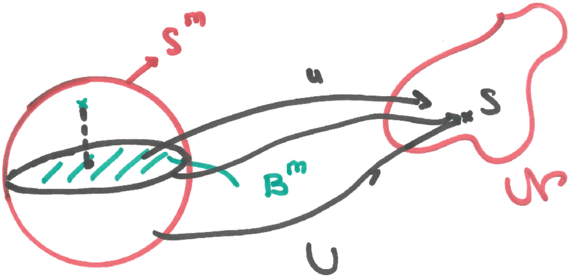
Any map $u \in C^0_{\mathrm{S}}(\mathbb{B}^m, \mathcal{N})$ yields a map $U \in C^0(\mathbb{S}^m, \mathcal{N})$ setting

$$\begin{cases} U(x_1, \dots, x_m, x_{m+1}) = u(x_1, \dots, x_m) \text{ if } x_{m+1} \geq 0 \\ \qquad\qquad\qquad\qquad\quad\;\; = S \text{ otherwise} \end{cases}$$

The obtained homotopy classes in $C^0_{\mathrm{S}}(\mathbb{B}^m, \mathcal{N})$ are then in one to one correspondance with that of $C^0(\mathbb{S}^m, \mathcal{N})$.



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U r

Remark

In some places, we will consider the set

$$C^0_{\mathrm{S}}(\mathbb{R}^m, \mathcal{N}) = \{u \in C^0(\mathbb{B}^m, \mathcal{N}), |u(x)| \to \mathrm{S} \text{ as } +\infty\}.$$

which is handled as $C^0_{\mathrm{S}}(\mathbb{B}^m, \mathcal{N}))$ so that we may define likewise homotopy classes.

We next give three important examples of homotopy classes, actually homotopy groups.

A first example: degree theory for maps from the circle to itself

Consider continuous maps from the circle $\mathbb{S}^1$ to $\mathbb{S}^1$ i.e. the set $C^0(\mathbb{S}^1, \mathbb{S}^1)$. For a map $u \in C^0(\mathbb{S}^1, \mathbb{S}^1)$, we may write

$$u(\exp i\theta) = \exp i\varphi, \text{ for } \theta \in [0, 2\pi[,$$

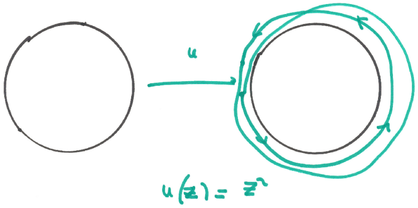
where $\varphi \in C^0([0, 2\pi], \mathbb{R})$ denotes the lifting of u, with

$$\varphi(2\pi) - \varphi(0) = 2k\pi \text{ where } k \in \mathbb{Z}$$

The number k counts the number of windings around the circle. It is a topological invariant i.e. it is the same in each homotopy class called the **degree** of the map u.

For instance the degree of the identity map $z \to z$ is 1
The degree of the map u given by $z \mapsto u(z) = z^2$ is 2, since

$$u(\exp i\theta) = \exp 2i\theta,$$



One may prove that two maps with the same degree are in the same homotopy class so that all homotopy classes are labelled by an integer, the degree. Hence, one has

$$\pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

Integral representation of the degree

Since $\varphi(2\pi) - \varphi(0) = 2k\pi$, one may write ($\times$ standing for the cross product in $\mathbb{R}^3$)

$$k = \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{d\theta} \varphi(\theta) \mathrm{d}\theta.$$

On the other hand, one has

$$\frac{d}{d\theta} \varphi(\theta) = u(\exp i\theta) \times \frac{d}{d\theta} u(\exp i\theta)$$

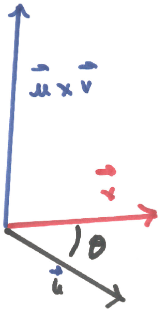
so that we are led to the integral formula

$$k = \frac{1}{2\pi} \int_{S^1} \left(u(s) \times \frac{d}{ds} u(s) \right) \mathrm{d}s.$$

which is very useful in the Sobolev framework !

The cross product

$$\begin{cases} |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta \\ \vec{u}\times\vec{v}\perp u, \vec{u}\times\vec{v}\perp v \end{cases}$$



As a side problem, we may ask, given a continuous map $u : \mathcal{M} \to \mathbb{S}^1$, if there exists a map $\varphi : \mathcal{M} \to \mathbb{R}$ such that

$$u(x) = \exp\left(i\varphi(x)\right), \forall x \in \mathcal{N}, \tag{Lifting}$$

The so-called lifting problem for $\mathbb{S}^1$ valued maps. We have

Theorem

Assume that $\pi_1(\mathcal{N}) = \{0\}$. Given any $u \in C^0(\mathcal{N}, \mathbb{S}^1)$, there exists a map $\varphi \in C^0(\mathcal{M}, \mathbb{R})$ such that the lifting property (Lifting) holds.

We given a short sketch of the proof, since it is rather simple.

Sketch of the proof

We introduce the one form $\alpha = u \times du = \sum u \times \partial_i u \mathrm{d}x_1$ so that

$$d\alpha = \sum (\partial_i u \times \partial_j u) \mathrm{d}x_i \wedge \mathrm{d}x_j = 0.$$

If $\mathcal{M}$ is simply connected, that it follows from Poincaré's Lemma that there exists a map φ such that

$$\alpha = d\varphi$$

We verify that φ has the desired property.

A degree theory can be developed for $\mathbb{S}^2$ valued maps in the same spirit as for $\mathbb{S}^1$ valued maps. It yields a topological invariant which classifies homotopy class of maps from $\mathbb{S}^2$ to $\mathbb{S}^2$. For a smooth $u : \mathbb{S}^2 \to \mathbb{S}^2$, it is given by the integral formula

$$\deg u = \int_{\mathbb{S}^2} u^\star(\omega_{_{\mathbb{S}^2}}) = \frac{1}{(4\pi)} \int_{\mathbb{S}^2} \det(\nabla u)\,\mathrm{d}x$$
$$= \frac{1}{(4\pi)} \int_{\mathbb{S}^2} u.u_{x_1} \times u_{x_2}\,\mathrm{d}x_1 \mathrm{d}x_2$$

where $\omega_{_{\mathbb{S}^3}}$ stands for a standard volume form on $\mathbb{S}^2$ and u^* stands for pullback.

Notice that the degree of the identity map of $\mathbb{S}^2$ is 1. The area formula yields a more geometrical interpretation, namely

$$\deg u = \sum_{a \in u^{-1}(z_0)} \mathrm{sign}(\det(\nabla u)),$$

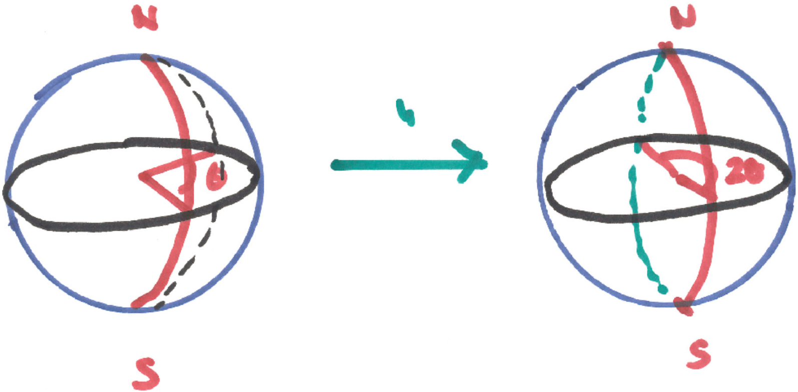
where $z_0 \in \mathbb{S}^2$ is any regular point, so that $u^{-1}(z_0)$ is a finite set.

The intuitive meaning is the number of times the map wraps the sphere.

Example The map u given by

$$u(r \exp i\theta, x_3) = u(r \exp i2\theta, x_3),\ 0 \leq r \leq 1, 0 \leq \theta \leq 1, r^2 + x_3^2 = 1,$$

has degree $+2$.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$\mathbb{S}^2$-valued maps with prescribed boundary

We describe an example of a degree one map with prescribed boundary.
et $\mathrm{S} = (0, 0-1)$ be the South pole of $\mathbb{S}^2$ and consider the set

$$C^0_{\mathrm{S}}(\mathbb{D}^2, \mathbb{S}^2) = \{u \in C^0(\mathbb{D}^2, S^2), u = S \text{ on } \partial D^2\}.$$

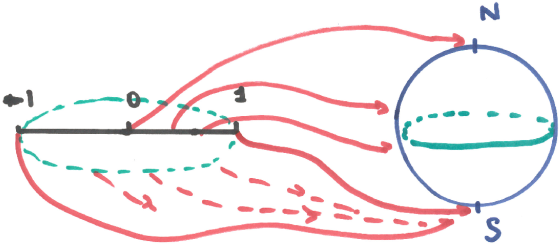
Homotopy classes in $C^0_{\mathrm{S}}(\mathbb{D}^2, \mathbb{S}^2)$ are labelled as before by the degree. Here is an example:

Let

$$\begin{cases} \chi(x_1, x_2) = (x_1 f(r), x_2 f(r), g(r)) \text{ with} \\ r = \sqrt{x_1^2 + x_2^2},\ r^2 f^2(r) + g^2(r) = 1, \end{cases}$$

with f and g smooth such that

$$\begin{cases} f(0) = f(1) = 0,\ 0 \le rf(r) \le 1 \text{ for any } r \in [0,1] \\ -1 \le g \le 1 \text{ and } g \text{ decreases from } g(0) = 1 \text{ to } g(1) = -1. \end{cases}$$



Then

$$\deg \chi = 1,$$

whereas the constant map $u = S$ has degree zero.

The Hopf invariant

This is a more sophisticated invariant.

It deals with continuous maps from $\mathbb{S}^3$ to $\mathbb{S}^2$, or $\mathbb{R}^3$ to $\mathbb{S}^2$, so that the counter-image of a regular point z_0 on the sphere $\mathbb{S}^2$ no longer is a point but instead a curve.

As for the degree, this invariant completely classifies the homotopy classes so that

$$\pi_3(\mathbb{S}^2) = \mathbb{Z}$$

as in thecase of the degree .

There are several definition of the invariant. The hopf map we present next is one way of understanding things.

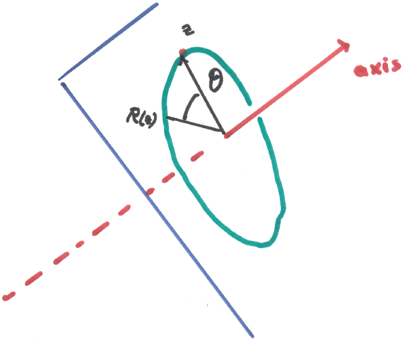
The Hopf map

We describe briefly the projection map $\Pi : \mathbb{S}^3 \to \mathbb{S}^2$ termed usually the Hopf map.

The sphere $\mathbb{S}^3$ is very close to the group of rotations $SO(3)$ of the three dimensional space $\mathbb{R}^3$, $\mathbb{S}^3$ may be in fact identified with its universal cover

Any rotation R in $SO(3)$ then yields an element on $\mathbb{S}^2$ considering the image by R of an arbitrary fixed point of the sphere, for instance the North pole $\mathrm{P}_{\mathrm{north}} = (0,0,1)$yielding a projection from $SO(3)$ to $\mathbb{S}^2$ considering the correspondance $R \mapsto R(P)$.

The construction of the projection Π from $\mathbb{S}^3$ onto $\mathbb{S}^2$ is readily in the same spirit, but requires to introduce some preliminary objects.



A rotation is characted by its axis, a line where points are left invariant, and an angle. The set of rotation living a point invariant is a circle.

Identifying $\mathbb{S}^3$ with $SU(2)$

$SU(2)$ denotes the group of two dimensional complex unitary matrices of determinant one, i.e. $SU(2) = \{U \in \mathbb{M}_2(\mathbb{C}),\ UU^* = I_2 \text{ and } \det(U) = 1\}$ or

$$SU(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix},\ \ a \in \mathbb{C}, b \in \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1 \right\} \simeq \mathbb{S}^3.$$

The Lie algebra of $SU(2)$ is the 3-dimensional space of traceless anti-hermitian matrices

$$su(2) = \{X \in \mathscr{M}_2(\mathbb{C}),\ X + X^* = 0 \text{ and } \mathrm{tr}(X) = 0\}$$

A canonical basis orthonormal for the euclidean norm $|X|^2 \equiv \det(X)$ is provided by the Pauli matrices

$$\sigma_1 \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \sigma_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_3 \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We identify the 2-sphere $\mathbb{S}^2$ with the unit sphere of $su(2)$

$$\mathbb{S}^2 \simeq \{X \in su(2),\ |X|^2 = \det(X) = 1\}.$$

The Hopf map. The group $SU(2)$ acts naturally on $su(2)$ by conjugation: if $g \in SU(2)$, then

$$su(2) \ni X \mapsto Ad_g(X) \equiv gXg^{-1} \in su(2).$$

Definition

The map $\Pi : SU(2) \simeq \mathbb{S}^3 \to \mathbb{S}^2 \subset su(2)$ defined, for $g \in SU(2)$ by

$$\Pi(g) \equiv Ad_g(\sigma_1) = g\sigma_1 g^{-1}$$

is called the Hopf map.

Notice that $\Pi(g) = \sigma_1$ if and only if g is of the form

$$g = \exp(\sigma_1 t) = \begin{pmatrix} \exp it & 0 \\ 0 & \exp -it \end{pmatrix} \text{ with } t \in [0, 2\pi]$$

If g and g' are such that $\Pi(g) = \Pi(g')$, then

$$g' = g \exp \sigma_1 t \quad \text{for some } t \in [0, 2\pi].$$

Hence the fiber $\Pi^{-1}(z)$ is diffeomorphic to the circle $\mathbb{S}^1$ for every $z \in \mathbb{S}^2$.

$SU(2)$ **appears hence as a fiber bundle with base space** $\mathbb{S}^2$ **and fiber** $\mathbb{S}^1$**.**

This bundle is not trivial since $Id_{\mathbb{S}^2}$ *does not* admit a continuous lifting $\Phi : \mathbb{S}^2 \to \mathbb{S}^3$ such that $Id_{\mathbb{S}^2} = \Pi \circ \Phi$. Indeed, Φ is homotopic to a constant, but not $Id_{\mathbb{S}^2}$.

Projecting maps onto $\mathbb{S}^2$

For a map $\mathrm{U} : \mathscr{M} \to \mathbb{S}^3$, we may associate to this map the map $u : \mathscr{M} \to \mathbb{S}^2$

$$u = \Pi \circ \mathrm{U}$$

The correspondan $\mathrm{U} \mapsto u$ is of course not one to one. Indeed, given any scalar function $\Theta : \mathscr{M} \to \mathbb{R}$, then we have

$$u = \Pi \circ \mathrm{U} = \Pi \circ \big(\mathrm{U} \exp(\sigma_1 \Theta(\cdot))\big).$$

Conversely, given two maps U_1 and U_2 such that $u = \Pi \circ \mathrm{U}_1 = \Pi \circ \mathrm{U}_2$ then there exists a map $\Theta : \mathscr{M} \to \mathbb{R}$ such that

$$\mathrm{U}_2 = \mathrm{U}_1 \exp(\sigma_1 \Theta(\cdot)).$$

The map Θ is referred to as the gauge freedom.

Lifting maps to $\mathbb{S}^2$ as maps to $\mathbb{S}^3$

Liftings correspond to "invert" Π: given $u : \mathcal{M} \to \mathbb{S}^2$, find $\mathrm{U} : \mathcal{M} \to \mathbb{S}^3$ s.t

$$u = \Pi \circ \mathrm{U}.$$

The map U is called a lifting of u. If a lifting exists due to gauges there is no uniqueness.

Important property

If $\mathcal{M}$ is simply connected, the lifting problem has always a solution in the **continuous class**, i.e. for any co $u \in C^0(\mathcal{M}, \mathbb{S}^2)$ there exists $\mathrm{U} \in C^0(\mathcal{M}, \mathbb{S}^3)$ such that

$$u = \Pi \circ \mathrm{U}.$$

Example. For $\mathcal{M} = \mathbb{S}^3$, the identity from $\mathbb{S}^3$ into itself is a lifting of the Hopf map.

Identifying homotopy classes

Sonce the lifting property holds in the continuous class allows to provide a one to one correspondance between homotopy classes in $C^0(\mathscr{M}, \mathbb{S}^3)$ and $C^0(\mathscr{M}, \mathbb{S}^2)$.

Indeed, two maps u_1 and u_2 from $\mathscr{M}$ to $\mathbb{S}^2$ are homotopic if and only if their respective liftings :coloturcU_1 and U_2 are in the same homotopy class (hint: deform gauges).

Specifying this property to $\mathscr{M} = \mathbb{S}^3$, we obtain an identification of $\pi_3(\mathbb{S}^2)$ and $\pi_3(\mathbb{S}^3)$ so that

$$\pi_3(\mathbb{S}^2) = \pi_3(\mathbb{S}^3) = \mathbb{Z}.$$

The Hopf Invariant

Given continous $u : \mathbb{S}^3 \to \mathbb{S}^2$ and a continuous lifting $\mathrm{U} : \mathbb{S}^3 \to \mathbb{S}^3$ such that

$$u = \Pi \circ \mathrm{U}$$

the degree theory for $\mathbb{S}^3$ valued maps allows to classify also the homotopy classes of maps from $\mathbb{S}^3$ to $\mathbb{S}^2$. We set

$$\mathrm{H}(u) = \deg(\mathrm{U}).$$

This number, is called the Hopf invariant of u it classifies homotopy classes in $C^0(\mathbb{S}^3, \mathbb{S}^2)$.

Example. Since $\Pi = \Pi \circ \mathrm{Id}_{\mathbb{S}^3}$, the Hopf invariant of the Hopf map Π is

$$\mathrm{H}(\Pi) = 1.$$

The homotopy class $[\Pi]$ is a generator of $\pi_3(\mathbb{S}^2)$.

Integral formulation of the Hopf Invariant

As for the degree, there exists a integral formulation of the Hopf invariant, which is however less direct.

$$\mathrm{H}(u) = \frac{1}{16\pi^2} \int_{\mathbb{S}^3} \alpha \wedge u^\star(\omega_{\mathbb{S}^2}), \text{ with } d\alpha = u^\star(\omega_{\mathbb{S}^2}),$$

α is not uniquely defined, we impose an additional gauge condition:

$$d^\star \alpha = 0, \text{ and hence } \alpha = d^\star \Phi, \tag{2}$$

for some two form ϕ. Since $\Delta = dd^\star + d^\star d$, we have the identity

$$\Delta_{\mathbb{S}^3} \Phi = u^\star(\omega),$$

and hence Φ is determined up to some additive constant form.

This leads to the definition

$$\mathrm{H}(u) = \frac{1}{16\pi^2}\int_{\mathbb{S}^3} d^\star\Phi \wedge u^\star(\omega_{_{\mathbb{S}^2}}), \text{ with } \Delta_{\mathbb{S}^3}\Phi = u^\star(\omega),$$

Notice now the order 4 dependance of this formula in terms of gradient.
Let us show next how this formula can be derived.

Let $\mathscr{M}$ be simply connected, $\mathrm{U}: \mathscr{M} \to SU(2)$ $u \equiv \Pi \circ \mathrm{U}$. We construct a 1-form A with values into the Lie algebra $su(2)$ setting

$$A \equiv \mathrm{U}^{-1} d\mathrm{U}.$$

Conversely, given any sufficiently smooth $su(2)$ valued 1-form A on $\mathscr{M}$ also called a connection, one may find a map $\mathrm{U}: \mathscr{M} \to SU(2)$ such that $A = \mathrm{U}^{-1} d\mathrm{U}$, iff

$$dA + \frac{1}{2}[A, A] = 0.$$

This is the zero curvature equation, a sort of nonlinear analog of Poincarés's Lemma.

Decomposing A on the basis of $su(2)$ as

$$A = A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_3,$$

we are led to

$$du = U[A, \sigma_1]U^{-1} = A_3\sigma_2 - A_2\sigma_3$$

A_2 and A_3 of A are completely determined by u, but notA_1 which is a gauge freedom. Indeed, for any sufficiently smooth function $\Theta : \mathscr{M} \to \mathbb{R}$, let $\mathrm{U}_\Theta(x) \equiv \exp(\Theta(x)\sigma_1)\mathrm{U}(x)$, so that $u = \Pi \circ \mathrm{U}_\Theta$ and

$$\mathrm{U}_\Theta^{-1} d\mathrm{U}_\Theta = \mathrm{U}^{-1} d\mathrm{U} + (d\Theta)\sigma_1 = A + d\Theta\sigma_1.$$

The values of A_2 and A_3 are left unchanged by the gauge transformation, and A_1 is changed into $A_1^\Theta = A_1 + d\Theta$. Moreover

$$\begin{cases} u^\star(\omega_{\mathbb{S}^2}) = A_2 \wedge A_3, \ \mathrm{U}^*(\omega_{\mathbb{S}^3}) = A_1 \wedge A_2 \wedge A_3, \\ |dU|^2 = |A_1|^2 + |A_2|^2 + |A_3|^2 \text{ and } |du|^2 = (|A_2|^2 + |A_3|^2), \end{cases} \tag{3}$$

where $\omega_{\mathbb{S}^2}$ stays for the standard volume form on $\mathbb{S}^2$.

The curvature equation yields

$$2dA_1 = A_2 \wedge A_3 = u^\star(\omega_{\mathbb{S}^2}),$$

so that dA_1 is determined u. Going back (3) we may write

$$\mathrm{U}^*(\omega_{\mathbb{S}^3}) = A_1 \wedge u^\star(\omega_{\mathbb{S}^2}).$$

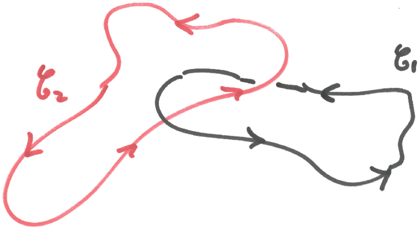
The integral formula for the degree of U yields in turn an integral formula for the Hopf invariant

$$\mathrm{H}(u) = \frac{1}{16\pi^2}\int_{\mathbb{S}^3} \alpha \wedge u^\star(\omega_{\mathbb{S}^2}), \text{ with } d\alpha = u^\star(\omega_{\mathbb{S}^2}),$$

where α corresponds to the one form $\alpha = 2A_1^\Theta$.

Linking numbers of curves and the Hopf Invariant

Another definition of the Hopf invariant is related to the notion of linking number of curves.



Definition and properties of the linking number

The linking number of two oriented curves $\mathscr{C}_1$ and $\mathscr{C}_2$ in $\mathbb{R}^3$ is given by the Gauss integral formula

$$\mathfrak{m}(\mathscr{C}_1, \mathscr{C}_2) = \frac{1}{4\pi} \oint_{\mathscr{C}_1} \oint_{\mathscr{C}_2} \frac{\overrightarrow{a_1 - a_2}}{|a_1 - a_2|^3} . \overrightarrow{da}_1 \times \overrightarrow{da}_2.$$

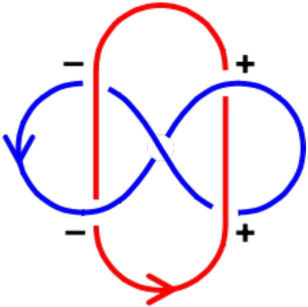
The linking number is always an integer, symmetric i.e.

$$\mathfrak{m}(\mathscr{C}_1, \mathscr{C}_2) = \mathfrak{m}(\mathscr{C}_2, \mathscr{C}_1),$$

its sign changes when the orientation of one of the curves is reversed and that $\mathfrak{m}(\mathscr{C}_2, \mathscr{C}_1) = 0$ if the two curves are **not linked**. In case of several connected components, we have

$$\mathfrak{m}(\mathscr{C}_{1,1} \cup \mathscr{C}_{1,2}, \mathscr{C}_2) = \mathfrak{m}(\mathscr{C}_{1,1}, \mathscr{C}_2) + \mathfrak{m}(\mathscr{C}_{1,2}, \mathscr{C}_2).$$

The linking number of two given curves can be computed as the half sum of the *signed crossing number* of a projection on a two dimensional plane.



Given a smooth map $u: \mathbb{R}^3 \to \mathbb{S}^2$ in $C_0^S(\mathbb{R}^3, \mathbb{S}^2)$ and a regular point M of $\mathbb{S}^2$, then its preimage $L_M \equiv u^{-1}(M)$ is a smooth bounded curve in $\mathbb{S}^3$. The main property is that the linking number $\mathfrak{m}(L_{M_1}, L_{M_2})$ of the preimages of any two regular points M_1 and M_2 on $\mathbb{S}^2$ is independent of the choice of the two points andequal to the Hopf invariant, that is

$$\mathfrak{m}(L_{M_1}, L_{M_2}) = \mathrm{H}(u).$$

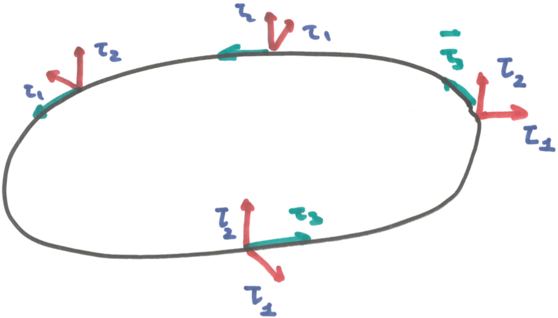
The Pontryagin Construction

This construction yields explicit examples of maps with prescribed Hopf number.

We start wit a **framed closed line** $\mathscr{C}$ in $\mathbb{R}^3$, i. e. which we are given a orthonormal basis of its orthogonal plane

$$\mathfrak{e}^{\perp}(\cdot) \equiv (\vec{\tau}_1(\cdot), \vec{\tau}_2(\cdot)).$$

This frame induces a natural *orientation* of the curve, choosing the vector $\vec{\tau}_3(a) = \vec{\tau}_1(a) \times \vec{\tau}_2(a)$ as a unit tangent vector, so that *any framed curve is oriented*.

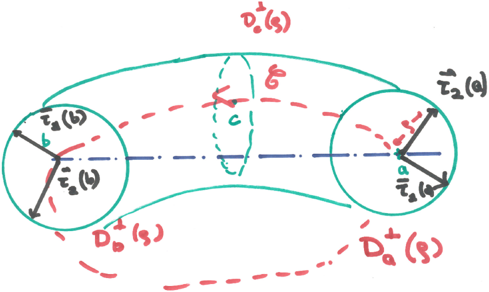


Our next task is to map a small annular region around the curve to the sphere $\mathbb{S}^2$. To that aim, we present first an elementary ingredient which is the construction of a map from a small disk onto the sphere $\mathbb{S}^2$.

Mapping a tubular neighborhood of $\mathscr{C}$ to $\mathbb{S}^2$

For $a \in \mathscr{C}$ let $P_a^\perp = (\mathbb{R}\vec{\tau}_a)^\perp \supset \mathbb{D}_a^\perp(\varrho)$, the disk centered at a of radius $\varrho > 0$. Set

$$\mathrm{T}_\varrho(\mathscr{C}) = \bigcup_{a \in \mathscr{C}} \mathbb{D}_a^\perp(\varrho)$$



We consider the map

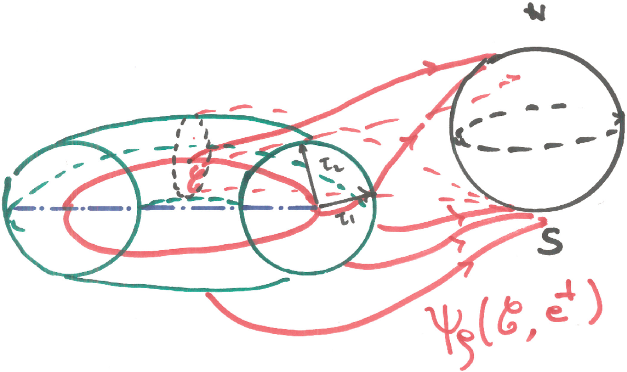
$$\begin{cases} \Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp](x_1\vec{\tau}_1(a) + x_2\vec{\tau}_2(a)) = \chi_\varrho(x_1, x_2) \text{ for} \\ x \equiv a + x_1\vec{\tau}_1(a) + x_2\vec{\tau}_2(a) \in \mathbb{D}^\perp_a(\varrho), \end{cases}$$

with $\chi_\varrho(x_1, x_2) = \chi(\varrho^{-1}x_1, \varrho^{-1}x_2)$.

This defines a smooth map $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp] : \mathrm{T}_\varrho(\mathscr{C}) \to \mathbb{S}^2$. Since $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp] = \mathrm{P}_{\mathrm{south}}$ on $\partial\mathrm{T}_r(\mathscr{C})$ we may extend this map to $\mathbb{R}^3$ setting

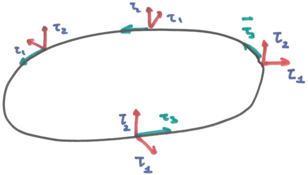
$$\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp](x) = S \text{ for } x \in \Omega_\varrho(\mathscr{C}) \equiv \mathbb{R}^3 \setminus \mathrm{T}_\varrho(\mathscr{C}),$$

so that $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$ is now a Lipschitz map from $\mathbb{R}^3$ to $\mathbb{S}^2$.



τ_2
τ_1
$\mathcal{C}$
S
$\Psi_\varrho(\mathcal{C}, e^{\perp})$

However, for a planar curve, with $\vec{\tau}_2$ perpendicular the Hopf invariant of $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$ is equal to zero.



To proof the map has trivial homotopy class,one may consider the linking number of preimages of any two regular points. We may consider as regular points the North pole P and the point M on the equator given by $M = (1,0,0)$. We have

$$L_{(\mathrm{P}_{\mathrm{north}})} = C \text{ whereas } L_M = \mathscr{C} + g^{-1}(0)\varrho\vec{e}_3,$$

where the function g is defined before. It follows that the two curves are parallel and hence nit linked so that in particular

$$\mathfrak{m}\left(L_{(\mathrm{P}_{\mathrm{north}})}, L_M\right) = 0.$$

The conclusion follows

Another proof of trivial homotopy class

We construct an explicit deformation with values into $\mathbb{S}^2$ of $\Psi_\varrho[\mathscr{C}, \mathfrak{e}_{\mathrm{ref}}^{\perp}]$ to a constant map.

The main step is to show that there exists a continuous map Φ from the exterior domain $\mathbb{R}^3 \setminus \mathscr{C}$ to $\mathbb{S}^1$ such that

$$\Phi(a + x_1 \vec{\tau}_1(a) + x_2 \vec{\tau}_2(a)) = \frac{(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}} \text{ for } a \in \mathscr{C} \text{ and } 0 < x_1^2 + x_2^2 \leq \varrho^2.$$

Assume Φ is given and define the deformation. We set, for $t \in [0,1]$

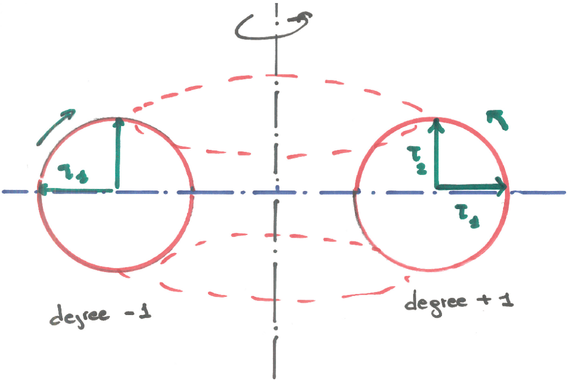
$$F(x,t) = \left((1-t) \frac{\mathfrak{r}(x)}{\varrho} f\left(\frac{\mathfrak{r}(x)}{\varrho}(1-t) \right) \Phi(x), g\left(\frac{\mathfrak{r}(x)}{\varrho}(1-t) \right) \right) \text{ for } x \in \mathbb{R}^3,$$

where f and g have been defined for χ and where $\mathfrak{r}$ is defined as

$$\begin{cases} \mathfrak{r}(x) = \sqrt{x_1^2 + x_2^2} \text{ for any } x = a + x_1 \vec{\tau}_1(a) + x_2 \vec{\tau}_2(a) \text{ with } a \in \mathscr{C}, 0 < x_1^2 + x_2^2 \leq \varrho^2, \\ \mathfrak{r}(x) = \varrho \text{ otherwise.} \end{cases}$$

Hence $F(\cdot, 0) = \mathrm{H}(\Psi_\varrho[\mathscr{C}, \mathfrak{e}_{\mathrm{ref}}^{\perp}])$ and $F(\cdot, 1) = \mathrm{P}_{\mathrm{north}}$, as desired.

The existence of Φ is easily seen in the axially symmetric case, using $\mathbb{S}^1$ degree theory.



Otherwise, use Biot and Savart.

There are two ways to create nontrivial Hopf invariants :

- Twisting the frame
- Linking two curves

Twisting the frame

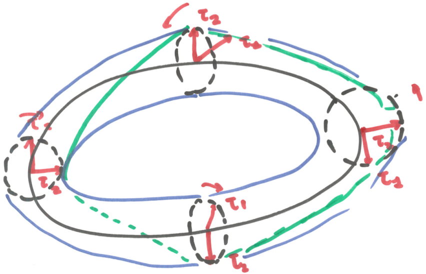
Consider a map $\gamma : \mathscr{C} \to SO(2) \simeq \mathbb{S}^1$, and consider the twisted frame

$$\mathrm{e}_\gamma^\perp = \gamma(\mathfrak{e}_{\mathrm{ref}}^\perp) \equiv (\gamma(\cdot)(\vec{\tau}_1(\cdot)), \gamma(\cdot)(\vec{\tau}_2(\cdot))),$$

where $\gamma(a)$ is considered as a rotation of the plane $(\tau_{\tan}(a))^\perp$. Since $\mathscr{C}$ is topologically equivalent to a circle, one may define a winding number of γ and prove, for instance using the crossing numbers, that

$$\mathrm{H}\left(\Psi_\varrho(\mathscr{C}, \mathrm{e}_\gamma^\perp)\right) = \deg(\gamma).$$

On the following example, the Hopf invariant is equal to 1.

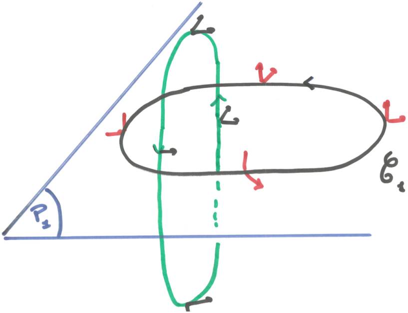


Notice that the blue and green lines are **linked**.

. . . or more generally the hopf invariant of $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$ is equal to the number d of twists.

Non trivial topology through linking

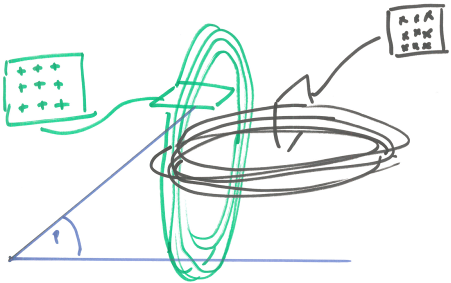
Another way to create maps with non trivial topology is to consider two linked planar curves, with non twisted frames



On the example the Hopf invariant of the map $\Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$ equals 2.

The k-Spaghetton map

Suppose now that we consider sheafs of k^2 curves which are linked as below



the Hopf invariant of the map $\mathfrak{S}_k \equiv \Psi_\varrho[\mathscr{C}, \mathfrak{e}^\perp]$, called the k spaghetton, equals now $2k^4$.