Branched transportation and singularities of Sobolev maps between manifolds
Part IV: branched transportation

Fabrice Bethuel

Laboratoire Jacques-Louis Lions,
Université Pierre et Marie Curie – Paris 6

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Optimal transportation to the boundary

This topic is of independent interest with some very intuitive features.

Branched transportation is a minimization problem which is involved in a very wide area of applications, including practical ones:

- computer science
- network design
- biology, for instance leafs growth.

It is also related to abstract question, as seen before in nonlinear functional analysis

Branched transportation appears when one seeks to optimize cost transportations, in the case the average cost of transportation decreases with density.

Consider a finite set A of points belonging to a bounded domain Ω of $\mathbb{R}^m$. The points in A have to be transported to the boundary $\partial\Omega$ of a domain, the cost functional involves the sum of the length of paths to the boundary multiplied by density function φ, depending on the density which represents the number of points using the same portion of paths joining to the boundary.

Since we are looking for minimizers, such paths are unions of segments, but possibly with variying densities. Branchings of the segments usually appear when the density function is **sublinear**. The intuitive idea is that it is cheaper to share the same path than to travel alone.

A typical example is given by the power law

$$\varphi(d) = d^{\alpha}, \text{ with given parameter } 0 < \alpha < 1,$$

so that φ is sublinear

$$(d_1 + d_2)^{\alpha} \leq d_1^{\alpha} + d_2^{\alpha} \text{ for large numbers}$$

and minimizers are expected to have segments with high multiplicity in order to decrease the total cost. They should also have several branching points, that is points in Ω where segments join in order to induce higher multiplicity.

The aim

Our aim is to describe the behavior of minimal branched connection when the number of points increases and ultimately goes to $+\infty$, assuming possibly the distribution of the points converges to some limiting finite measure μ, i.e.

$$\frac{1}{(\sharp A)}\sum_{a\in A}\delta_a \to \mu$$

Special emphasis will be put on the case the measure μ is proportional to the Lebesgue measure : it is the case when points are equidistributed, on grids for instance.

The critical exponent

As we will see, the value

$$\alpha_m = 1 - \frac{1}{m},$$

termed the critical exponent is central.

- The case $1 > \alpha > \alpha_m$ has been studied thoroughly in the existing litterature: it has been shown by Xia that one may define a limiting minimizing problem for the measure μ.
- In constrast, less results are known for $0 \leq \alpha \leq \alpha_m$, in particular the construction of a limiting problem for the measure μ remains widely open.

The book by Bernot, Caselles and Morel is an excellent presentation of this field.

Mathematical framework

The:colo theory of graphs is well adapted to the kind of objects we wish to describe. It involves

- **Points**. These are of two kinds: the points we wish to connect to the boundary, but also additional points which are th branching points and points on the boundary
- **Oriented segments**. they join the previously mentioned points. Orientation is important.

Assume that we are given :

- a lipschitz open bounded domain $\Omega \subset \mathbb{R}^m$, $m \in \mathbb{N}^*$
- a finite set $A = \{a_1, \dots, a_l\} \subset \overline{\Omega}$. This set corresponds later to a set of positive charges +1, or sources.

The aim is to model **directed graphs connecting** the set A to the boundary $\partial\Omega$

We closely follows the presentation of the seminal work of **[Xia, 03]**, adapted to the case of connection to the boundary.

A directed graph G is represented by the following data:

- a finite vertex set $V(G) \subset \overline{\Omega}$.We assume furthermore that:

$$A \subset V(G) \subset \overline{\Omega}, \text{ **i.e. sources are vertices.**}$$

- A set $E(G)$ of **directed segments** joining the vertices, possibly with multiplicity: For $e \in E(G)$, we denote by e^- and e^+ the **endpoints** of e.

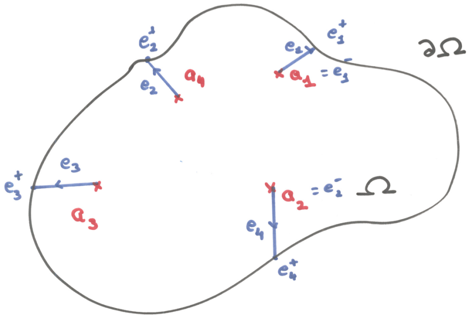
For $a \in V(G)$, set $E^{\pm}(a, G) = \{e \in E(G), e^{\pm} = a\}$. We impose for $a \in V(G) \setminus \partial\Omega$ the Kirchhoff law

$$(Kirchhoff) \qquad \begin{cases} \sharp(E^-(a,G)) = \sharp(E^+(a,G)) + 1 \text{ if } a \in A \\ \sharp(E^-(a,G)) = \sharp(E^+(a,G)) \text{ if } a \in V(G) \cap \Omega \setminus A, \end{cases}$$

and impose moreover that if $[e^-, e^+] \in E(G)$ then $[e^+, e^-] \notin E(G)$, and denote by $\mathscr{G}(A, \partial\Omega)$ the set of directed graphs with the previous properties.

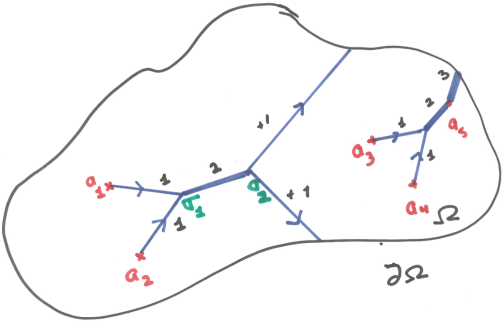
Some graphs in $\mathscr{G}(A, \partial\Omega)$

Segments joining the boundary: The simplest example is provided by segments joining the points a_i to the boundary $\partial\Omega$.



In this case $\mathbf{e_i} = [a, e_i^+]$, with $e_i^+ \in \partial\Omega$.

Segments with multiplicities



On this picture $V(G) \cap \Omega = \{a_1, \ldots, a_5, \sigma_1, \sigma_2\}$, the points σ_i having charge 0. It follows from Kirchhoff's law that there is a segment with multiplicity 2 and one with multiplicty 3.

Multiplicities of segments

For graphs in $G \in \mathcal{G}(A, \partial\Omega)$ that there exists a *unique* multiplicity function $d : E(G) \to \mathbb{N}$ such that, if $a \in V(G) \setminus \partial\Omega$

$$\sum_{e \in E(G), e^- = a} d(e) = \underset{\substack{e \in E(G) \\ e^+ = a}}{d}(e) + \begin{cases} 1 \text{ if} & a \in A \\ 0 \text{ otherwise} \end{cases}$$

Moreover

$$\sum_{e^+ \in \partial\Omega} d(e) - \sum_{e^- \in \partial\Omega} d(e) = \sharp(A),$$

It follows in particular that, if A is non-empty, then $V(G) \cap \partial\Omega \neq \emptyset$.

The functional and minimal branched connections to the boundary

Given $0 \leq \alpha \leq 1$, consider the functional W_α defined on $\mathscr{G}(A, \partial\Omega)$

$$\mathrm{W}_\alpha(G) = \sum_{e \in E(G)} (d(e))^\alpha \mathscr{H}^1(e) \quad \text{for } G \in \mathscr{G}(A, \partial\Omega). \tag{1}$$

and then the non-negative quantity

$$\mathfrak{L}^\alpha_{\mathrm{brbd}}(A, \partial\Omega) = \inf\{\mathrm{W}_\alpha(G), G \in \mathscr{G}(A, \partial\Omega)\}, \tag{2}$$

the branched connection of order α of the set A to the boundary $\partial\Omega$.

Important observation : For $0 \leq \alpha < 1$

(subadditivity) $$(d_1 + d_2)^\alpha \ll d_1^\alpha + d_2^\alpha$$

Hence, better to have high multiplicities

High implicities $\Rightarrow$ branching points.

Notice that :

- $\alpha = 1$ corresponds to optimal transportation of the points to the boundary.
- $\alpha = 0$ to minimal graphs connecting the set A to the boundary.

Other simple observations:

- as a consequence of subadditivity

$$(LB) \qquad \mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A, \partial\Omega) \geq \mathrm{dist}(A, \partial\Omega) \sharp(A)^{\alpha}.$$

- Worst case is when points are far appart from each others (e.g. on a grid).

For $\alpha > \alpha_m = 1 - \frac{1}{m}$, where $m = \dim \Omega$ the lower bound (LB) corresponds also somehwat to an upper-bound for well-distributed sets of points.

An important observation made by Xia is :

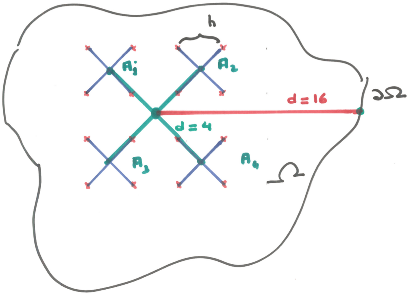
Proposition (Xia, 03)

Assume that $\alpha \in (\alpha_m, 1]$, *where* $\alpha_m = 1 - \frac{1}{m}$. *Then we have*

$$\mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A, \partial\Omega) \leq C(\Omega, \alpha)(\sharp(A))^{\alpha}.$$

The proof is :

- obvious in the case $\alpha = 1$.
- obtained through a dyadic type decomposition in the general case. Ideas are related to self similarity and multiscale phenomena



Setting $N = \sharp A$, $h \simeq N^{-\frac{1}{m}}$, we derive

$$\mathrm{W}_\alpha(G) \propto h.N + (\frac{N}{2^m}).2h.(2^m)^\alpha + \ldots$$

$$\propto h.N\left(1 + 2^\gamma + \ldots + 2^{k\gamma}\right) \text{suming up to } 2^k h \simeq 1.$$

where $\gamma = 1 + m(\alpha - 1)$. $\alpha \leq \alpha_m$ then $\gamma > 0$, and the estimate follows in the case considered.

Similar arguments show that, if $0 \leq \alpha < \alpha_m = 1 - \frac{1}{m}$ then

$$\mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A, \partial\Omega) \leq C(\Omega, \alpha)\left(\sharp(A)\right)^{\alpha_m} (\text{ instead of } \left(\sharp(A)\right)^{\alpha}).$$

and that if $\alpha = \alpha_m$ then

$$\mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A, \partial\Omega) \leq C(\Omega) \log\left(\sharp(A)\right)\left(\sharp(A)\right)^{\alpha_m}.$$

We turn next to **upper bounds** and restrict ourselves to points **on regular grids**.

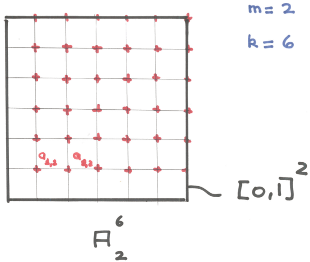
Remark: Notice that dimension plays an important role in the definition of the critical exponent α_m.

The case uniform grids

Here Ω is unit cube $\Omega = [0,1]^m$ and the points of A are on a grid.
Let k in $\mathbb{N}^*$, set $h = \frac{1}{\mathrm{k}}$ and consider

$$A_m^{\mathrm{k}} \equiv \boxplus_m^{\mathrm{k}}(h) = \left\{ a_{\mathrm{I}}^k \equiv h I = h(i_1, i_2, \dots, i_m), \text{ for } I \in \{1, \dots, \mathrm{k}\}^m \right\},$$

so that $\sharp(A_m^{\mathrm{k}}) = \mathrm{k}^m$.



We set

$$\Lambda_m^\alpha(\mathrm{k}) = \mathfrak{L}^\alpha_{\mathrm{brbd}}(A_m^{\mathrm{k}}, \partial(0,1)^m) \text{ and } \Xi_m^\alpha(\mathrm{k}) \equiv \mathrm{k}^{-m\alpha}\Lambda_m^\alpha(\mathrm{k})$$

and study the behavior of $\Lambda_m^\alpha(\mathrm{k})$ and $\Xi_m^\alpha(\mathrm{k})$ as $\mathrm{k} \to +\infty$. For $\alpha < \alpha_m$ recall the upper bound

$$\Lambda_m^\alpha(\mathrm{k}) \le C_\alpha \mathrm{k}^{m\alpha} \quad \text{i.e.} \quad \Xi_m^\alpha(\mathrm{k}) \le C_\alpha,$$

In the critical case $\alpha = \alpha_m = 1 - \frac{1}{m}$, this upper bound no longer holds.

Theorem (B, 14')

We have the lower bound

$$\Lambda_m^{\alpha_m}(\mathrm{k}) \geq \mathrm{C}_m \mathrm{k}^{m\alpha_m} \log \mathrm{k} = \mathrm{C}_m \mathrm{k}^{m-1} \log \mathrm{k},$$

that is

$$\Xi_m^{\alpha_m}(\mathrm{k}) \geq \mathrm{C}_m \log \mathrm{k}.$$

1 The lower bound does not hold for an arbitrary measure μ. For instance if all points are localized at the center of $\Omega = \mathbb{B}^m$, then $\mu = \delta_0$ since

$$\frac{1}{(\sharp A)} \sum_{a \in A} \delta_a \to \delta_0$$

In this case we have

$$\mathfrak{L}^{\alpha_m}_{\mathrm{brbd}}(A, \partial\Omega) \leq C(\Omega, \alpha)(\sharp(A))^{\alpha_m}.$$

so that there is no log divergence.

Remark

2. The fact that the renormalized quantity

$$\Xi_m^{\alpha_m}(\mathrm{k}) = \mathrm{k}^{1-m}\Lambda_m^{\alpha_m}(\mathrm{k})$$

does not remain bounded as $\mathrm{k} \to +\infty$ is related to the fact that the Lebesgue measure *is not irrigible* for the critical value $\alpha = \alpha_m$, a result which has been proved By Devillanova and Solimini (see also the book by Bernot, Morel and Caselles).

We next present a few ideas in the proof.

sketch of the proof of the lower- bound for a grid

We are going to make use again of self-similarity properties of the functional.

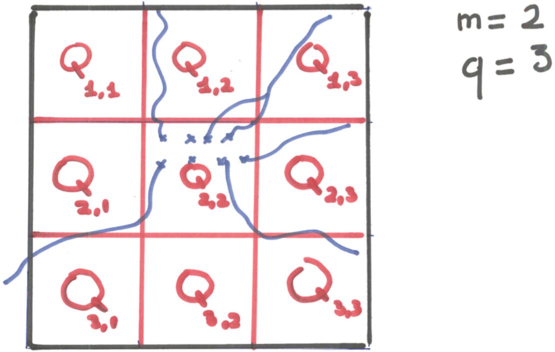
We introduce for that purpose an arbitrary integer paramater $\mathfrak{q} > 1$ We notice, for arbitrary exponent α, the scaling property

$$\mathfrak{L}^{\alpha}_{\mathrm{brbd}}(\frac{1}{\mathfrak{q}}A^k_m, \partial(0, \frac{1}{\mathfrak{q}})^m) = \mathfrak{q}^{-1}\Lambda^{\alpha}_m(\mathrm{k}), \text{ for } \mathfrak{q} \in \mathbb{N}^*.$$

We decompose $[0,1]^m$ as an union of $\mathfrak{q}^m$ disjoint smaller cubes

$$[0,1]^m = \underset{\mathfrak{p}\in\mathfrak{P}}{\cup} \overline{\mathrm{Q}}_{\mathfrak{p}}, \text{with } \mathfrak{p} \equiv (j_1, j_2, \ldots, j_m) \in \mathfrak{P} = \{0, \ldots, \mathfrak{q}-1\}^m,$$

where $\mathrm{Q}_{\mathfrak{p}} = \frac{1}{\mathfrak{q}}(j_1, j_2, \ldots, j_m) + (0, \frac{1}{\mathfrak{q}})^m$.


$Q_{1,1}$
$Q_{1,2}$
$Q_{1,3}$
$Q_{2,1}$
$Q_{2,2}$
$Q_{2,3}$
$Q_{3,1}$
$Q_{3,2}$
$Q_{3,3}$
$m = 2$
$q = 3$

Setting

$$A_{\mathfrak{p}}^{\mathrm{q}} \equiv A_m^{\mathrm{qk}} \cap \mathrm{Q}_{\mathfrak{p}},$$

so that $A_{\mathfrak{p}} = \frac{1}{\mathrm{q}}\mathfrak{p} + \frac{1}{\mathrm{q}} A_m^k$. We have $\mathfrak{L}_{\mathrm{brbd}}^{\alpha}(A_{\mathfrak{p}}, \partial \mathrm{Q}_{\mathfrak{p}}) = \mathrm{q}^{-1} \Lambda_m^{\alpha}(\mathrm{k})$.

This leads a first elementary scaling law

$$\Xi_m^{\alpha}(\mathrm{qk}) \geq k^{m(\alpha_m - \alpha)} \Xi_m^{\alpha}(\mathrm{k}), \quad \text{for any } \mathrm{k} \in \mathbb{N}^*.$$

This scaling property can be improved as follows

Lemma

Let $\mathfrak{q} \in \mathbb{N}^*$ *be given. There exists some constant* $\mathrm{C}^{\alpha}_{\mathfrak{q}} > 0$ *such that*

$$\Xi^{\alpha}_{m}(\mathfrak{q}\,\mathrm{k}) \geq \mathfrak{q}^{m(\alpha_m - \alpha)}\,\Xi^{\alpha}_{m}(\mathrm{k}) + \mathrm{C}^{\alpha}_{\mathfrak{q}}, \quad \text{for any } \mathrm{k} \in \mathbb{N}^*.$$

This then leads in the case $\alpha = \alpha_m$ to the inequality of the Theorem

$$\Xi^{\alpha_m}_{m}(\mathrm{k}) \geq \mathrm{C}_m \log \mathrm{k}.$$

Remark. In the case $\alpha < \alpha_m$ we obtain the bound

$$\Lambda^{\alpha}_{m}(\mathrm{k}) \geq C_{\alpha}\, k^{m\alpha_m} \text{ so that } \Lambda^{\alpha}_{m}(\mathrm{k}) \propto k^{m\alpha_m}.$$

Summarizing, we obtain

- $\Lambda_m^\alpha(\mathrm{k}) \propto k^{m\alpha}$ for $\alpha > \alpha_m$
- $\Lambda_m^\alpha(\mathrm{k}) \propto k^{m\alpha_m}$ for $\alpha < \alpha_m$
- $\Lambda_m^\alpha(\mathrm{k}) \propto (\log k)k^{m\alpha_m}$ for $\alpha = \alpha_m$

Some open questions

- Address the previous limiting problems in the spirit of Gamma convergence.
  This is done in the case $\alpha > \alpha_m$ [Xia, Morel, Solimini,...].
  Open for $\alpha \leq \alpha_m$. It seems that a completely new approach is required for in the case $\alpha > \alpha_m$
- study **maximal density for optimal graphs** $G^{\alpha}_{\rm opt}$.
  In order to quantify branching, it is naturel to turn to the density function and to consider for a given graph $G \in \mathcal{G}(A, \partial\Omega)$ the quantity

$$d_{\rm max}(G) = \sup\{d(e), e \in V(G)\},$$

which represents the highest density of segments inside G. A large number $d_{\rm max}$) indicates the presence of multiple branchings.

With respect to the density question, a very intuitive result is the following, based on the fact that optimal graphs have no loops:

Lemma

We have $d_{\max}(G^{\alpha}_{\mathrm{opt}}) \leq \sharp(A)$.

One may conjecture that

$$d_{\max}(G^{\alpha}_{\mathrm{opt}}) \propto \sharp(A)$$

for all values of a α.Known to be true for $\alpha > alpha_m$.

back to nonlinear functional analysis

Recall that our our main motivation was sequentially weak density of smooth maps in Sobolev spaces $W^{1,p}(\mathcal{M},\mathcal{N})$, $\mathcal{M},\mathcal{N}$ being manifolds
Strong density well understood for some time : iff $\pi_{[p]}(\mathcal{N}) \neq \{0\}$.

The only open case is given by the following:
Open case: $1 \leq p < m$, $\pi_p(\mathcal{N}) \neq \{0\}$, and p **is an integer**.

The answer depends crucially on further properties of $\mathcal{N}$.
The first is $\mathcal{N} = \mathbb{S}^p$.

Theorem (B-Zheng 88, B 91)

Let p be an integer. Then given any manifold $\mathcal{M}$, $C^\infty(\mathcal{M},\mathbb{S}^p)$ is sequentially weakly dense in $W^{1,p}(\mathcal{M},\mathbb{S}^p)$.

In constrast, we will show in the case $p=3$ and $\mathscr{N}=\mathbb{S}^2$:

Theorem (B 14)

Given any manifold $\mathscr{M}$ of dimension larger then 4, $C^\infty(\mathscr{M},\mathbb{S}^2)$ is **not sequentially weakly dense** *in $W^{1,3}(\mathscr{M},\mathbb{S}^2)$.*

Strongly related to properties of the Hopf Fibration. Perhaps more surprising at first sight with **optimal transportation**.

We will first consider the case $p = 3$, $\mathscr{M} = \mathbb{B}^4$ and show that, in that case there exist maps in

$$W_{\mathrm{S}}^{1,3}(\mathbb{B}^4, \mathbb{S}^2) = \{u \in W^{1,3}(\mathbb{B}^4, \mathbb{S}^2), u(x) = \mathrm{S} \text{ for } \mathrm{x} \in \partial \mathrm{B}^4\}$$

that are **NOT** weak limits of smooth maps.

Proposition

There exists a map $\mathscr{U}$ is $W_{\mathrm{S}}^{1,3}(\mathbb{B}^4, \mathbb{S}^2)$ which is not the weak limit of smooth maps between $\mathbb{B}^4$ and $\mathbb{S}^2$.

Elements in the construction

Recall that set of maps with a finite number of isolated singularities

$$\mathscr{R}(\mathbb{B}^4,\mathbb{S}^2)=\{u\in W^{1,3}(\mathbb{B}^4,\mathbb{S}^2),\ \text{s.t } u\in C^\infty(\mathbb{B}^m\setminus\{A\}) \text{ for a finite set } A\}.$$

of Hopf number ± 1 is **dense** in $W^{1,3}(\mathbb{B}^4,\mathbb{S}^2)$

Defect measure

Let $u \in \mathscr{R}(\mathbb{B}^4, \mathbb{S}^2)$. Consider the set $\mathcal{V}(u)$ of all sequences $(v_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^4, \mathbb{S}^2)$ such that

$$v_n \rightharpoonup u \text{ weakly in } W^{1,3}(\mathbb{B}^m, \mathbb{S}^2) \text{ and}$$

the first observation is

Lemma

The set of sequences $\mathcal{V}(u)$ is not empty.

The proof amounts to construct explicitly a sequence $(v_n)_{n\in\mathcal{N}}$ which converges weakly to u. This can be done attaching "lines of bubbles" to u in order to remove the singularities.

Optimal sequences

If one seeks for an optimal sequence from the point of view of energy, then are led to consider the number

$$\mu^\star(u) = \inf_{(v_n)_{n\in\mathcal{N}}\in\mathcal{V}} \left\{ \limsup_{n\to+\infty} \mathrm{E}_3(v_n) \right\}$$

Theorem (Hardt-Rivière 03)

We have

$$\mu^\star(u) = \mathrm{E}_3(u) + \Gamma(A),$$

with

$$\Gamma(A) \geq C\mathrm{L}_{\mathrm{branch}}(A),$$

where A denotes the set of singularities of U and $\mathrm{L}_{\mathrm{branch}}(A)$ represents branched transportation of parameter $\alpha = \frac{3}{4} = \alpha_4$ joining positive singularities to negative singularities of A or to the boundary.

The following is the main ingredient in the construction of the map $\mathscr{U}$:

Lemma

Given any $k \in \mathbb{N}^*$, there exists a map $\mathfrak{v}_k \in \mathscr{R}_{\mathrm{S}}(\mathbb{B}^4, \mathbb{S}^2)$ such that

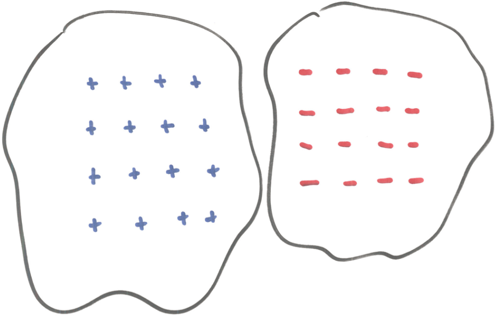
$$\begin{cases} \mathrm{E}_3(\mathfrak{v}_k) \leq \mathrm{C}_1 k^3, \mathrm{C}_1 > 0 \\ \mathrm{L}_{\mathrm{branch}}(\mathfrak{v}_k) \geq \mathrm{C}_2 \log(k) k^3, \ \mathrm{C}_2 > 0 \end{cases}$$

The functional $\mathrm{L}_{\mathrm{branch}}(v) = \mathrm{L}_{\mathrm{branch}}(A)$ refers to a branched transportation with exponent $\frac{3}{4}$ connecting singularities of opposite of A signs or to the boundary.

$$\text{defect energy} \simeq \mathrm{L}_{\mathrm{branch}}(\mathfrak{v}_k) \geq C(\log k)\,\mathrm{E}_3(\mathfrak{v}_k),$$

comments

The function $\mathfrak{v}_k$ of the Lemma has k^4 singularities of charge $+1$, as well as k^4 singularities of charge -1. These $+1$ are located on a uniform grid, with distance between nearest neighbors of order k^{-1}, far from the negative charges as on the picture below.



Using the branched transportation results

Since we have to join the postive charges to negative ones or to the boundary, junctions have to cross the border of the domain containing postive charges, so that

$$\mathrm{L}_{\mathrm{branch}}(v_k) \geq \mathfrak{L}^{\alpha}_{\mathrm{brbd}}(A_k, \partial\Omega),$$

with $\sharp(A_k) = k^4$ so that

$$\log(\sharp(A_k))(\sharp(A_k))^{\frac{3}{4}} \propto (\log k)k^3$$

hence the estimate for $\mathrm{L}_{\mathrm{branch}}(v_k)$.

The key Lemma yields counter-examples to weak density

The map $\mathscr{U}$ described in the main theorem above is obtained:

- pasting a infinite countable number of copies of scaled and translated versions of the maps $\mathfrak{v}_k$ for suitable choices of the integer k and the scaling factors.
- This gluing is performed in such a way that the energies sum up to provide a finite total energy whereas the values for the respective functional $\mathrm{L}_{\mathrm{branch}}$ do not: this is made possible since the two quantities behave differently as k grows.
- The conclusion then immediately follows from the convergence by Hardt and Rivière.

Thank you for your attention!