Sobolev maps between manifolds and branched transportation.

Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, with $\mathcal{N}$ isometrically embedded in $\mathbb{R}^\ell$, we consider the Sobolev spaces

$$W^{1,p}(\mathcal{M},\mathcal{N}) = \{u \in W^{1,p}(\mathcal{M},\mathbb{R}^\ell),\ u(x) \in \mathcal{N} \text{ for almost every } x \in \mathcal{M}\}, \quad \text{for } 1 \leq p \leq +\infty.$$

In order to describe properties of these maps, a central issue is the approximation of maps in $W^{1,p}(\mathcal{M},\mathcal{N})$ by smooth maps between $\mathcal{M}$ and $\mathcal{N}$ in the case $1 \leq p < m \equiv \dim \mathcal{M}$, the answer being affirmative if $p \geq m$ due to Sobolev embedding. Whereas strong approximability quite well understood, it can be proved for instance that if $\mathcal{M}$ is a ball, then strong approximability holds if and only if $\pi_{[p]}(\mathcal{N}) \neq 0$, $[p]$ denoting the largest integer less or equal to p. On the other hand, weak approximability, which is the main focus of the lectures, remains quite widely open in the case$p$ is an integer.

We will start with the critical case $p = m$, where the bubbling phenomenon yield a first example of the concentration phenomena, which is at the heart of the discussion. For $p < m$, we then concentrate on the simple case $\mathcal{N} = \mathbb{S}^2$, and discuss two cases which have been treated successfully so far. First, we will show that weak approximability holds when $p = 2$, for which $\pi_2(\mathbb{S}^2) = \mathbb{Z}$ is related to degree theory. We then turn to the case $p = 3$, for which one has likewise $\pi_3(\mathbb{S}^2) = \mathbb{Z}$, but for which sequentially weak approximability *does not hold*, the later case being related to the Hopf invariant. The construction of counterexamples rely on specific properties of this invariant combining various aspects of three-dimensional topology, in particular linking numbers of curves through the Pontryagin construction. In both cases, the results are connected to *optimal transportation*, point singularities being removed when $m = p + 1$ using concentration along lines. The case $p = 2$ corresponds to standard mass transportation, whereas the case $p = 3$ corresponds to *branched transportation with exponent* $\alpha = 3/4$. This exponent is critical for the *irrigation of the Lebesgue measure* in dimension 4, an issue that will be discussed also.